Difference Schemes on Uniform Grids Performed by General Discrete Operators

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Abstract

Our aim is to set the foundations of a discrete vectorial calculus on uniform n-dimensional grids, that can be easily re-formulated on general irregular grids. As the key tool we first introduce the notion of tangent space to any grid node. Then we define the concepts of vector field, field of matrices and inner products on the space of grid functions and on the space of vector fields, mimicking the continuous setting. This allows us to obtain the discrete analogous of the basic first order differential operators, gradient and divergence, whose composition define the fundamental second order difference operator. As an application, we show that all difference schemes, with constant coefficients, for first and second order differential operators with constant coefficients can be seen as difference operators of the form $-\mathsf{div}\,(\mathsf{A}\nabla u)+\langle \mathsf{b}, \nabla u\rangle + q\,u$ for suitable choices of q, b and A. In addition, we characterize special properties of the difference scheme, such as consistency, symmetry and positivity in terms of q, b and A.

1 Introduction

We develop here a discrete vectorial calculus on grids of the Euclidean n-dimensional space to obtain the difference operators that are the discrete analogous of second order complete differential operators with constant coefficients. So, our work would be considered in the framework of mimetic discretizations and hence parallels to those developed by Samarski's school, *i.e.*, the method of support-operators or mimetic difference schemes, [7, 8, 9, 11]. As they, we construct the operational calculus from a basic operator, the gradient in our case. But unlike them, we define *a priori* the concept of tangent space to a grid node. Once both notions have been introduced, the concepts of vector field, matrix field, and in particular the concept of metric tensor, appear in a natural way. Analogously to the continuous setting, we define the divergence operator as the negative of the gradient adjoint, with respect to the canonical inner products. So, by composing the gradient operator with a field of matrices and also with the divergence, we obtain the fundamental difference second order operator. This, can be considered as the Laplace-Beltrami operator of the grid when the field of matrices is a metric tensor. In any case, the expressions of these discrete operators in the grid nodes are formally equal to that of the difference schemes used to approximate linear second order differential operators. Conversely, we will prove that any difference scheme can be seen as a difference operator.

We set our focus on uniform grids, however the concepts and techniques here presented are in force for arbitrary grids. The simple structure of the underlying space highlights the role played by the discrete vector and matrix fields. A formulation on general networks was developed by the authors in [1] which includes the Green formulae and a wide treatment of self-adjoint boundary value problem.

We principally focus on characterizing structural properties of difference schemes such as symmetry or consistency, in terms of the vector and matrix fields. In addition, we will pay special attention to the schemes of positive type, that lead to linear systems whose coefficient matrix is a strictly diagonally dominant M-matrix, that is a strictly diagonally dominant matrix with non positive off diagonal coefficients. These matrices are very suitable to solve systems by iterative methods (see [3, 6, 12].)

As no initial boundary value problem for a linear differential equation is raised in this work, we will not tackle the search of conditions that assures the convergence of the solutions of discrete problems to the solutions of corresponding continuous ones. However, it must be observed that the mimetic formulations allow to prove discrete conservation laws that imply the stability of the discretization and hence the convergence, provided that the consistency of the discretization is ensured. In addition, the *supraconvergence* results for mimetic discretizations obtained by J.M. Hyman and S. Steinberg in [10] show the interest of considering also difference schemes with low consistency order.

Here we will deal with consistent schemes with general second or first order linear differential operators with constant coefficients in $\mathbb{R}^n$, that is, those of the form

$$L(u) = -\sum_{i,j=1}^{n} k_{ij} u_{x_i x_j} + \sum_{j=1}^{n} k_j u_{x_j} + k_0 u,$$

where $k_{ij} = k_{ji} \in \mathbb{R}$, $i,j = 1 \dots n$, $k_j \in \mathbb{R}$, $j = 0, \dots, n$ verify that $\sum\limits_{i,j=1}^{n} |k_{ij}| + \sum\limits_{j=1}^{n} |k_j| > 0$. Alternatively, if we consider the symmetric matrix $K = (k_{ij})$ and the vector $\mathsf{k} = (k_1, \dots, k_n)$, the operator L can be rewritten as

$$L(u) = -\mathsf{div}\,(K\nabla u) + \langle \mathsf{k}, \nabla u\rangle + k_0 u.$$

The operator L is *selfadjoint* iff $\mathsf{k} = \mathsf{0}$ and it is called *elliptic* or *semi-elliptic* when K is a definite or semidefinite matrix, respectively. In this case, we can suppose without loss of generality that K is positive definite or semidefinite.

If $r \in \mathbb{Z}$, we will say that a function $\alpha\colon (0, +\infty) \longrightarrow \mathbb{R}$ is *of order* r if there exists $C > 0$ such that $|\alpha(h)| \le Ch^r$ for all h small enough. For each $r \in \mathbb{Z}$, $O(h^r)$ denote the vectorial spaces of all functions of order r. It is clear that $O(h^r) \subset O(h^s)$ for each $s < r$ and $O(h^r)O(h^s) \subset O(h^{r+s})$. If $A\colon (0,+\infty) \longrightarrow M_{k\times m}(\mathrm{I\!R})$, is given by $A(h) = \Big(a_{ij}(h)\Big)$, we say that $A \in O(h^r)$ iff $a_{ij} \in O(h^r)$, $i = 1,\dots,k$, $j = 1,\dots,m$.

2 Difference schemes

Fix $n \in \mathbb{N}^*$, for each $h > 0$ we consider the subset in $\mathrm{I\!R}^n$ given by $V_h = h\mathbb{Z}^n$. The nodes $x, y \in V_h$ are called *neighbors* if their euclidean distance $|x - y|$ equals h, and that will be geometrically represented by means of the segment that joint them, s_{xy}. The set of neighbors to x will be denoted by $V_h(x)$.

We will call *n-dimensional uniform grid of size h*, Γ_h, the set V_h together with the above neighborhood relation.

To build difference schemes on the family of grids $\{\Gamma_h\}_{h>0}$, we will use for each $h > 0$ and any $x \in V_h$ a *stencil* or computational molecule, $S_h(x)$, containing at least the node x and its neighbors, that is $\{x\}, V_h(x) \subset S_h(x)$. The set $S_h(x) \setminus \{x\}$ will be denoted by $S'_h(x)$.

A *difference scheme on* $\{\Gamma_h\}_{h>0}$ is an expression of the form

$$L_h(u)(x) = \gamma_{xx}(h)u(x) - \sum_{y\in S'_h(x)} \gamma_{xy}(h)u(y), \quad x \in V_h, \quad \gamma_{xy}(h)\colon (0,+\infty) \longrightarrow \mathrm{I\!R},$$

where u is an arbitrary function on $\mathrm{I\!R}^n$, or equivalently,

$$L_h(u)(x) = q_x(h)\, u(x) + \sum_{y\in S'_h(x)} \gamma_{xy}(h)\,\Big(u(x) - u(y)\Big), \quad x \in V_h \tag{1}$$

where $q_x(h) = \gamma_{xx}(h) - \sum\limits_{y\in S'_h(x)} \gamma_{xy}(h)$.

The difference scheme L_h is called *r-consistent with the differential operator L on* $\{\Gamma_h\}_{h>0}$ if there exists $r > 0$ such that $L(u)(x) - L_h(u)(x) = O(h^r)$, $x \in V_h$ for any u smooth enough; see for instance [12] and [13]. The number r is called *order of consistency of the scheme* and functions $\gamma_{xy}, y \in S_h(x)$ are named *coefficients of the scheme.* For the sake of simplicity in the sequel the expression *r-consistent scheme* will mean a difference scheme consistent with the operator L on V_h and we will omit the argument h in the coefficients of the scheme.

In what follows, for all $x \in V_h$ we will consider the stencil $S_h(x)$ that contains the node x, its neighbors and also the neighbors to the neighbors to x, that is, $S_h(x) = \{x\} \bigcup V_h(x) \bigcup\limits_{y\in V_h(x)} V_h(y)$.

Therefore, if $\{\mathsf{e}_j\}_{j=1}^n$ denotes the standard basis of $\mathrm{I\!R}^n$ and we define $\mathsf{e}_{n+j} = -\mathsf{e}_j$, $j = 1,\dots,n$, then for any $x \in V_h$, the stencil $S_h(x)$ is formed by x and the following nodes:

$$x_j = x + h\mathsf{e}_j, \quad j = 1,\dots,2n \quad \text{and} \quad x_{ij} = x + h\,(\mathsf{e}_i + \mathsf{e}_j), \quad 1 \le i \le j \le 2n, \quad j \ne n+i.$$

It is clear that $V_h(x) = \{x_j\}_{j=1}^{2n}$ for any $x \in V_h$ and that $|S_h(x)| = 2n(n+1)+1$. Figure 1 displays the bidimensional stencil $S_h(x)$.

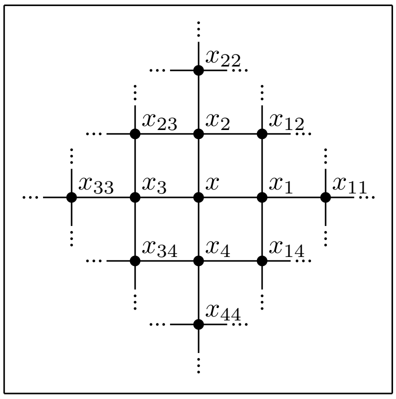

Figure 1: Bidimensional stencil

Here we will only deal with *schemes with constant coefficients*. This means that there exist functions γ_j, $j = 0, \ldots, 2n$ and γ_{ij}, $1 \le i \le j \le 2n$, $j \ne n+i$, such that for all $x \in V_h$

$$\gamma_0 = \gamma_{xx}, \quad \gamma_j = \gamma_{xx_j}, \quad j = 1, \ldots, 2n, \qquad \gamma_{ij} = \gamma_{xx_{ij}}, \quad 1 \le i \le j \le 2n, \quad j \ne n+i.$$

So, the scheme (1) can be rewritten as

$$L_h(u)(x) = q\, u(x) + \sum_{j=1}^{2n} \gamma_j \Big(u(x) - u(x_j)\Big) + \sum_{\substack{1 \le i \le j \le 2n \\ j \ne n+i}} \gamma_{ij} \Big(u(x) - u(x_{ij})\Big), \qquad x \in V_h, \tag{2}$$

where $q = \gamma_0 - \sum_{j=1}^{2n} \gamma_j - \sum_{\substack{1 \le i \le j \le 2n \\ j \ne n+i}} \gamma_{ij}$.

The scheme L_h will be called *quasi-symmetric* if there exists h_0 such that for all $0 < h \le h_0$ it is satisfied that $\gamma_{ij}(h) = \gamma_{n+in+j}(h)$ for all $1 \le i \le j \le n$ and $\gamma_{jn+i}(h) = \gamma_{in+j}(h)$ for all $1 \le i < j \le n$. The scheme L_h will be called *symmetric* if it is quasi-symmetric and $\gamma_j(h) = \gamma_{n+j}(h)$, for all $j = 1, \ldots, n$.

The scheme L_h is called *of non negative type* if there exists h_0 such that for all $0 < h \le h_0$, $q(h) \ge 0$, $\gamma_j(h) \ge 0$, $j = 1 \ldots, 2n$ and $\gamma_{ij}(h) \ge 0$ for each $1 \le i \le j \le 2n$, $j \ne n+i$. The scheme L_h is called *of positive type* if it is of non negative type and there exists $C > 0$ such that $h^2 \gamma_j(h) \ge C$, for all $j = 1, \ldots, 2n$ and for h small enough.

Of course L_h would be consistent if the coefficients of the scheme (2) verified some conditions. As usual these conditions are obtained by replacing in (2) the values of u at the nodes of the stencil $S'_h(x)$ by its Taylor expansion. For this, it will be useful to add to our terminology the functions

$\gamma_{ji} = \gamma_{ij}$, $1 \le i \le j \le 2n$, $j \ne n+i$. So, for a fixed integer $m \ge 1$ and for each $x \in V_h$ we obtain

$$\begin{aligned} L(u)(x) - L_h(u)(x) = {} & \phi^0\, u(x) + \sum_{k=1}^{m} \frac{h^k}{k!} \left[\sum_{j=1}^{n} \phi_j^k\, D_j^k u(x) \right. \\ & + \left. \sum_{l=1}^{k-1} \binom{k}{l} \sum_{1 \le i < j \le n} \psi_{ij}^{lk-l}\, D_i^l D_j^{k-l} u(x) \right] + T_{m+1}(x), \end{aligned}$$

where functions ϕ^0, ϕ_j^k, for all $j = 1, \dots, n$ and ψ_{ij}^{lk-l}, for all $1 \le i < j \le n$, $k = 2, \dots, m$ and $l = 1, \dots, k$ are given by the following equalities:

$$\begin{aligned} \phi^0 &= k_0 - q, \\ \phi_j^1 &= \gamma_j - \gamma_{n+j} + 2(\gamma_{jj} - \gamma_{n+jn+j}) + \sum_{\substack{i=1 \\ i \ne j}}^{n} (\gamma_{ij} - \gamma_{n+in+j} + \gamma_{jn+i} - \gamma_{in+j}) + \frac{k_j}{h} \end{aligned} \tag{3}$$

$$\begin{aligned} \phi_j^2 &= \gamma_j + \gamma_{n+j} + 4(\gamma_{jj} + \gamma_{n+jn+j}) + \sum_{\substack{i=1 \\ i \ne j}}^{n} (\gamma_{ij} + \gamma_{n+in+j} + \gamma_{jn+i} + \gamma_{in+j}) - \frac{2k_{jj}}{h^2}, \\ \psi_{ij}^{11} &= \gamma_{ij} + \gamma_{n+in+j} - \gamma_{jn+i} - \gamma_{in+j} - \frac{2k_{ij}}{h^2}. \end{aligned} \tag{4}$$

and for each $k = 3, \dots, m$,

$$\begin{aligned} \phi_j^k &= \gamma_j - \gamma_{n+j} + 2^k(\gamma_{jj} - \gamma_{n+jn+j}) + \sum_{\substack{i=1 \\ i \ne j}}^{n} (\gamma_{ij} - \gamma_{n+in+j} + \gamma_{jn+i} - \gamma_{in+j}), \quad k \text{ odd} \\ \phi_j^k &= \gamma_j + \gamma_{n+j} + 2^k(\gamma_{jj} + \gamma_{n+jn+j}) + \sum_{\substack{i=1 \\ i \ne j}}^{n} (\gamma_{ij} + \gamma_{n+in+j} + \gamma_{jn+i} + \gamma_{in+j}), \quad k \text{ even} \end{aligned} \tag{5}$$

$$\psi_{ij}^{lk-l} = \begin{cases} \gamma_{ij} + \gamma_{n+in+j} + \gamma_{jn+i} + \gamma_{in+j}, & \text{if } k = 2r,\ l = 2s,\ s = 1, \dots, r-1, \\ \gamma_{ij} + \gamma_{n+in+j} - \gamma_{jn+i} - \gamma_{in+j}, & \text{if } k = 2r,\ l = 2s-1,\ s = 1, \dots, r, \\ \gamma_{ij} - \gamma_{n+in+j} - \gamma_{jn+i} + \gamma_{in+j}, & \text{if } k = 2r-1,\ l = 1, \dots, r-1, \\ \gamma_{ij} - \gamma_{n+in+j} + \gamma_{jn+i} - \gamma_{in+j}, & \text{if } k = 2r-1,\ l = r, \dots, 2(r-1). \end{cases} \tag{6}$$

The function $T_{m+1}(x)$ depends linearly on the functions $h^2\gamma_j$ and $h^2\gamma_{ij}$, $i, j = 1, \dots, 2n$, $|i-j| \ne n$. In addition, when $m = 1$, function $T_2(x)$ also depends on the second order coefficients of L, $\{k_{ij}\}_{i,j=1}^n$. Therefore, to study consistency properties related with second order differential operators we must consider $m \ge 2$.

Proposition 2.1 *Consider $m \geq 1$ and suppose that $K = 0$ when $m = 1$. Then, the following conditions are verified*

$$\gamma_k,\ \gamma_{ij},\ \in O(h^s), \quad k = 1,\dots,2n, \quad 1 \leq i \leq j \leq 2n, \quad j \neq n+i,$$

$$\phi^0 \in O(h^r),\ \phi_j^k \in O(h^{r-k}), \quad k = 1,\dots,m, \quad j = 1,\dots,n,$$

$$\psi_{ij}^{lk-l} \in O(h^{r-k}) \quad k = 2,\dots,m, \quad l = 1,\dots,k-1, \quad 1 \leq i < j \leq n,$$

where $r > 0$ and $s > -(m+1)$. Then, the scheme (2) is consistent of order $\min\{r, s+m+1\}$.

From equations (3) and (4), if $m = 1$ necessarily $s \leq -1$, whereas if $m \geq 2$, necessarily $s \leq -2$ when $K \neq 0$. Therefore, when $K = 0$, that is when L is a first order differential operator, to find consistent schemes of order 1 it suffices to consider $m = 1$, whereas to find consistent schemes of higher order it is necessary to choose $m \geq 2$. On the other hand, when $K \neq 0$, that is when L is a second order differential operator, to find consistent schemes of order 1 it suffices to consider $m = 2$, whereas to find higher order consistent schemes it is necessary to choose $m \geq 3$.

3 Difference operators

Our main goal in this section is to develop a difference calculus on the uniform grid of size h, Γ_h. For that we consider Γ_h as a discrete manifold and we proceed by analogy with the continuous case. The key concept will be the tangent space at a node of the grid.

We will denote by $\mathcal{C}(V_h)$ the set of real functions defined on V_h and if $F \subset V_h$, by $\mathcal{C}(F)$ the subset of $\mathcal{C}(V_h)$ formed by the functions that vanish on $V_h \setminus F$. If $u \in \mathcal{C}(V_h)$, the *support of* u is the set $\mathsf{supp}(u) = \{x \in V_h : u(x) \neq 0\}$. Moreover, the set of real functions on V_h whose support is a finite subset will be denoted by $\mathcal{C}_0(V_h)$.

For any $x \in V_h$, we define the *tangent space at* x as the vectorial space $T_x(\Gamma_h)$ of the formal linear combinations of the segments incident with x. Therefore, system $\{s_{xy}\}_{y \in V_h(x)}$ is a basis of $T_x(\Gamma_h)$ and the elements of $T_x(\Gamma_h)$ are of the form $\mathsf{v} = \sum\limits_{y \in V_h(x)} v_y s_{xy}$, where $v_y \in \mathbb{R}$ for each $y \in V_h(x)$. Note that $\dim T_x(\Gamma_h) = 2n$ for any $x \in V_h$.

A *vector field* is an application that assigns to each node a vector of its tangent space. So, if f is a vector field, then for each $x \in V_h$, $\mathsf{f}(x) = \sum\limits_{y \in V_h(x)} f(x,y) s_{xy}$. Therefore, a vector field can be represented by means of its *component function*, that is, the function $f \colon V_h \times V_h \longrightarrow \mathbb{R}$ such that for each $x \in V_h$, $f(x,y) = 0$ if $y \notin V_h(x)$. We will denote by $\mathcal{X}(\Gamma_h)$ the set of vector fields on Γ_h. If $\mathsf{f} \in \mathcal{X}(\Gamma_h)$ the *support of* f is $\mathsf{supp}(\mathsf{f}) = \{x \in V_h : \mathsf{f}(x) \neq \mathsf{0}\}$. The set of fields whose support is a finite subset will be denoted by $\mathcal{X}_0(\Gamma_h)$.

If f is a vector field we will say that f is *non negative*, and it will be represented by $\mathsf{f} \geq \mathsf{0}$, iff its component function, f, is non negative. Moreover, we will denote by $|\mathsf{f}|$ the non negative vector field whose component function is $|f|$. In addition, if $\mathsf{f},\mathsf{g}\in\mathcal{X}(\Gamma)$, the inequality $\mathsf{f}\geq\mathsf{g}$ will mean that $\mathsf{f}-\mathsf{g}\geq 0$.

A vector field f is called a *flow* if its component function f satisfies that $f(x,y)=-f(y,x)$ for all $x,y\in V_h$. If $\mathsf{f}\in\mathcal{X}(\Gamma_h)$ we will call the *flow determined by* f the vector field $\hat{\mathsf{f}}$ whose component function is given by $\hat{f}(x,y)=\frac{1}{2}\Big(f(x,y)-f(y,x)\Big)$ for all $x,y\in V_h$, where f stands for the component function of f. It is clear that if $\mathsf{f}\in\mathcal{X}_0(\Gamma_h)$, then $\hat{\mathsf{f}}\in\mathcal{X}_0(\Gamma_h)$.

We will say that the vector field f is *homogeneous* if there exist a vector $b=(b_j)\in\mathbb{R}^{2n}$ such that $f(x,x_j)=b_j$ for all $x\in V_h$, $j=1,\dots,2n$. In this case we say that the homogeneous field f is *determined by the vector* b. Moreover, $|\mathsf{f}|$ is the homogeneous vector field determined by $(|b_1|,\dots,|b_{2n}|)$ and f is non negative iff $b_j\geq 0$.

Note that when f is a homogeneous field determined by $b\in\mathbb{R}^{2n}$, then f is a flow iff $b_{n+j}=-b_j$, $j=1,\dots,n$. In this case we say that f is the *homogeneous flow determined by* $(b_1,\dots,b_n)\in\mathbb{R}^n$. Moreover, if f is the homogeneous field determined by b, then $\hat{\mathsf{f}}$ is the homogeneous flow determined by $\hat{b}=(\hat{b}_j)$ where $\hat{b}_j=\frac{1}{2}\,(b_j-b_{n+j})$, $j=1,\dots,n$.

If $\mathsf{f},\mathsf{g}\in\mathcal{X}(\Gamma_h)$ and f,g are their component functions, the expression $\langle\mathsf{f},\mathsf{g}\rangle$ will denote the function belonging to $\mathcal{C}(V_h)$ given by $\langle\mathsf{f},\mathsf{g}\rangle(x)=\sum\limits_{y\in V_h(x)}f(x,y)g(x,y)$, $x\in V_h$. This function allows the definition of the inner product on the space $\mathcal{X}_0(\Gamma_h)$ determined by

$$\frac{1}{2}\sum_{x\in V_h}\langle\mathsf{f},\mathsf{g}\rangle(x)\equiv\frac{1}{2}\int_{V_h}\langle\mathsf{f},\mathsf{g}\rangle(x)\,dx,\qquad \mathsf{f},\mathsf{g}\in\mathcal{X}_0(\Gamma_h)\tag{7}$$

where factor $\frac{1}{2}$ is due to the fact that each segment joining adjacent vertices is taken into account twice. We also consider the standard inner product on $\mathcal{C}_0(V_h)$ defined by

$$\sum_{x\in V_h}u(x)v(x)\equiv\int_{V_h}u(x)v(x)\,dx,\qquad u,v\in\mathcal{C}_0(V_h).\tag{8}$$

Both inner products will be the basic tools for the construction of the operational calculus. In addition, expressions (7) and (8) have also sense when only one of the fields or functions, respectively, have finite support.

A *field of matrices on* Γ_h is an application A that assigns to each node $x\in V_h$ a matrix $\mathsf{A}(x)$ of order $\dim T_x(\Gamma_h)$. Therefore, if A is a field of matrices there exist functions $a_{ij}\in\mathcal{C}(V_h)$, $i,j=1,\dots,2n$, called *component functions of* A, such that $\mathsf{A}(x)=\Big(a_{ij}(x)\Big)$, for each $x\in V_h$. We will say that A is a *field of non-singular matrices* if for each $x\in V_h$ the matrix $\mathsf{A}(x)$ is non-singular. In this case we denote by A^{-1} the field of inverse matrices of A.

We will say that the field of matrices A is *diagonal, symmetric or positive definite* if for each $x\in V_h$ the matrix $\mathsf{A}(x)$ is diagonal, symmetric or positive definite, respectively. The field A is a *metric tensor* if it is a field of symmetric and positive definite matrices. Moreover, if A is a metric tensor then the basis $\{s_{xy}\}_{y\in V_h(x)}$ of $T_x(\Gamma_h)$ is orthogonal for all $x\in V_h$ iff A is a diagonal field.

We will say that the field of matrices A is *isotropic* if there exist functions $a_1, a_2, a_3 \in \mathcal{C}(V_h)$ such that $a_{jj}(x) = a_1(x)$, $a_{jn+j}(x) = a_{n+jj}(x) = a_2(x)$, for all $j = 1, \dots, n$ and $a_{ij}(x) = a_3(x)$, otherwise.

We will say that the field of matrices A is *homogeneous* if its component functions are constant.

It is clear that if A is a field of non-singular matrices, then it is homogeneous or isotropic iff the same happens to the field A^{-1}. Moreover, if A is a metric tensor, then it is non-singular and hence A^{-1} is also a metric tensor.

If A is the homogeneous field of matrices determined by A, we will denote by d_{A} and r_{A} the homogeneous vector fields determined by the diagonal elements of A and the vector formed by the row sums of A, respectively. Conversely, if f is the homogeneous vector field determined by f, then D_{f} will stand for the homogeneous field of matrices determined by the diagonal matrix whose diagonal entries are given by f.

In the case we are dealing with the grid has a uniform structure, so we are only interested in vector and matrix fields that agree with this structure. For this reason, we will only consider homogeneous vector fields and homogeneous fields of matrices on Γ_h. Specifically, from now on b stands for the homogeneous vector field determined by b, $b = (b_j) \in \mathbb{R}^{2n}$ and A stands for the homogeneous field of matrices determined by A, where $A = (a_{ij})$ is a matrix of order $2n$.

To start with the operational calculus we will take the gradient as the basic operator and we will deduce the rest of operators by means of duality and composition techniques, as it is usual in the continuous setting.

The *gradient operator* assigns to each $u \in \mathcal{C}(V_h)$ the vector field ∇u determined by the expression

$$h\nabla u(x) = \sum_{j=1}^{2n} \Big(u(x_j) - u(x)\Big) s_{xx_j}, \qquad x \in V_h \tag{9}$$

where s_{xx_j} is the basis for the tangent space at x. Observe that $h\nabla$ is given by an incidence matrix. Moreover, $\nabla u \in \mathcal{X}_0(\Gamma_h)$ when $u \in \mathcal{C}_0(V_h)$ and ∇u is always a flow. In addition, $\nabla u = 0$ iff u is a constant function.

The *divergence operator* assigns to each $\mathsf{f} \in \mathcal{X}(\Gamma_h)$ the function $\operatorname{div} \mathsf{f} \in \mathcal{C}(V_h)$ determined by the relation

$$\int_{V_h} u(x) \operatorname{div} \mathsf{f}(x)\, dx = -\frac{1}{2} \int_{V_h} \langle \mathsf{f}, \nabla u \rangle (x)\, dx, \qquad \text{for each } \; u \in \mathcal{C}_0(V_h). \tag{10}$$

Thus, if f is the component function of f, then

$$\operatorname{div} \mathsf{f}(x) = \frac{1}{2h} \sum_{j=1}^{2n} \Big(f(x, x_j) - f(x_j, x)\Big), \qquad x \in V_h. \tag{11}$$

It is clear that $\operatorname{div} \mathsf{f}(x) = \dfrac{1}{h} \displaystyle\sum_{j=1}^{2n} \hat{f}(x, x_j)$, for all $x \in V_h$, which implies that $\operatorname{div} \mathsf{f} = \operatorname{div} \hat{\mathsf{f}}$. Moreover, $\operatorname{div} \mathsf{f} \in \mathcal{C}_0(V_h)$ when $\mathsf{f} \in \mathcal{X}_0(\Gamma_h)$ and the divergence can be formally defined as $\operatorname{div} = -\nabla^*$ on $\mathcal{X}_0(\Gamma_h)$, that is, as the negative of the adjoint operator of the gradient with respect to the inner products defined on $\mathcal{C}_0(V_h)$ and $\mathcal{X}_0(\Gamma_h)$. In addition, identity (10) holds when $\mathsf{f} \in \mathcal{X}_0(\Gamma_h)$ and $u \in \mathcal{C}(V_h)$.

We also consider the operators $\mathsf{A}\nabla$ and $\langle \mathsf{b}, \nabla\rangle$ that assign to each function $u \in \mathcal{C}(V_h)$ the vector field $\mathsf{A}\nabla u$ and the function $\langle \mathsf{b}, \nabla u\rangle$, determined respectively by

$$\begin{aligned} \mathsf{A}\nabla u(x) &= \frac{1}{h}\sum_{i=1}^{2n}\Big[\sum_{j=1}^{2n} a_{ij}\Big(u(x_j)-u(x)\Big)\Big] s_{xx_i}, \qquad && x \in V_h, \\ \langle \mathsf{b}, \nabla u\rangle(x) &= \frac{1}{h}\sum_{i=1}^{2n} b_j\Big(u(x_j)-u(x)\Big), && x \in V_h. \end{aligned} \tag{12}$$

Of course, $\mathsf{A}\nabla u \in \mathcal{X}_0(\Gamma_h)$ and $\langle \mathsf{b}, \nabla u\rangle \in \mathcal{C}_0(V_h)$ when $u \in \mathcal{C}_0(V_h)$. Moreover, if b is a homogeneous flow, then

$$\langle \mathsf{b}, \nabla u\rangle(x) = \frac{1}{h}\sum_{i=1}^{n} b_j\Big(u(x_j)-u(x_{n+j})\Big), \qquad x \in V_h. \tag{13}$$

The difference operators, ∇, div, $\mathsf{A}\nabla$ and $\langle \mathsf{b}, \nabla\rangle$, are all *first order difference operators*, in the sense that for each $u \in \mathcal{C}(V_h)$ or $\mathsf{f} \in \mathcal{X}(\Gamma_h)$ and for each $x \in V_h$, $\nabla u(x)$, $\mathsf{A}\nabla u(x)$, $\langle \mathsf{b}, \nabla u\rangle(x)$ and $\mathsf{div}\,\mathsf{f}(x)$ only take into account the values of u or f at the node x and at nodes in $V_h(x)$. In the same way, a (difference) operator on $\mathcal{C}(V_h)$ or $\mathcal{X}(\Gamma_h)$ will be called a *second order difference operators* if for each $x \in V_h$ the values of the image function or of the image field only depend on the nodes of the stencil $S_h(x)$. Of course, the composition of two first order difference operators produces a second order difference operator.

Our next objective is to present the fundamental second order difference operator on Γ_h. Specifically, we deal with the operator $\mathsf{div}\,(\mathsf{A}\nabla u)$. Observe that when A is a metric tensor then operators $\mathsf{A}\nabla$ and $\mathsf{div}\,(\mathsf{A}\nabla)$ can be considered respectively as the gradient and the *Laplace-Beltrami* operators, with respect to the metric tensor A^{-1}.

Proposition 3.1 *If $u \in \mathcal{C}(V_h)$ then for each $x \in V_h$*

$$
\begin{aligned}
-\mathsf{div}\,(\mathsf{A}\nabla u)(x) &= \frac{1}{2h^2}\sum_{j=1}^{n}\Big(\sum_{i=1}^{2n}(a_{ij}+a_{n+ji})\Big)\Big(u(x)-u(x_j)\Big)\\
&+\frac{1}{2h^2}\sum_{j=1}^{n}\Big(\sum_{i=1}^{2n}(a_{ji}+a_{in+j})\Big)\Big(u(x)-u(x_{n+j})\Big)\\
&-\frac{1}{2h^2}\sum_{j=1}^{n}a_{n+jj}\Big(u(x)-u(x_{jj})\Big)-\frac{1}{2h^2}\sum_{j=1}^{n}a_{jn+j}\Big(u(x)-u(x_{n+jn+j})\Big)\\
&-\frac{1}{2h^2}\sum_{1\le i<j\le n}(a_{n+ij}+a_{n+ji})\Big(u(x)-u(x_{ij})\Big)\\
&-\frac{1}{2h^2}\sum_{1\le i<j\le n}(a_{in+j}+a_{jn+i})\Big(u(x)-u(x_{n+in+j})\Big)\\
&-\frac{1}{2h^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(a_{ij}+a_{n+jn+i})\Big(u(x)-u(x_{jn+i})\Big)
\end{aligned}
$$

Proof. If f denotes the component function of $\mathsf{A}\nabla u$, then for each $x \in V_h$

$$\mathsf{div}\,(\mathsf{A}\nabla u)(x) = \frac{1}{2h}\sum_{i=1}^{2n}\Big(f(x,x_i)-f(x_i,x)\Big).$$

When $i = 1,\ldots,2n$, then $f(x,x_i) = \dfrac{1}{h}\displaystyle\sum_{j=1}^{2n}a_{ij}\Big(u(x_j)-u(x)\Big)$ and therefore

$$\sum_{i=1}^{2n}f(x,x_i) = \frac{1}{h}\sum_{j=1}^{2n}\Big(\sum_{i=1}^{2n}a_{ij}\Big)\Big(u(x_j)-u(x)\Big).$$

On the other hand, if $i = 1,\ldots,n$ then

$$
\begin{aligned}
f(x_i,x) &= \frac{1}{h}\sum_{j=1}^{i-1}\Big[a_{n+ij}\Big(u(x_{ji})-u(x_i)\Big)+a_{n+in+j}\Big(u(x_{in+j})-u(x_i)\Big)\Big]\\
&+\frac{1}{h}\Big[a_{n+ii}\Big(u(x_{ii})-u(x_i)\Big)+a_{n+in+i}\Big(u(x)-u(x_i)\Big)\Big]\\
&+\frac{1}{h}\sum_{j=i+1}^{n}\Big[a_{n+ij}\Big(u(x_{ij})-u(x_i)\Big)+a_{n+in+j}\Big(u(x_{in+j})-u(x_i)\Big)\Big]\\
&=\frac{1}{h}\Big(\sum_{j=1}^{2n}a_{n+ij}\Big)\Big(u(x)-u(x_i)\Big)+\frac{1}{h}a_{n+ii}\Big(u(x_{ii})-u(x)\Big)\\
&+\frac{1}{h}\sum_{j=1}^{i-1}a_{n+ij}\Big(u(x_{ji})-u(x)\Big)+\frac{1}{h}\sum_{j=i+1}^{n}a_{n+ij}\Big(u(x_{ij})-u(x)\Big)\\
&+\frac{1}{h}\sum_{j=1}^{i-1}a_{n+in+j}\Big(u(x_{in+j})-u(x)\Big)+\frac{1}{h}\sum_{j=i+1}^{n}a_{n+in+j}\Big(u(x_{in+j})-u(x)\Big)
\end{aligned}
$$

and

$$\begin{aligned}
f(x_{n+i},x) &= \frac{1}{h}\sum_{j=1}^{i-1}\Big[a_{ij}\Big(u(x_{jn+i})-u(x_{n+i})\Big)+a_{in+j}\Big(u(x_{n+jn+i})-u(x_{n+i})\Big)\Big] \\
&\quad+\frac{1}{h}\Big[a_{ii}\Big(u(x)-u(x_{n+i})\Big)+a_{in+i}\Big(u(x_{n+in+i})-u(x_{n+i})\Big)\Big] \\
&\quad+\frac{1}{h}\sum_{j=i+1}^{n}\Big[a_{ij}\Big(u(x_{jn+i})-u(x_{n+i})\Big)+a_{in+j}\Big(u(x_{n+in+j})-u(x_{n+i})\Big)\Big] \\
&=\frac{1}{h}\Big(\sum_{j=1}^{2n}a_{ij}\Big)\Big(u(x)-u(x_{n+i})\Big)+\frac{1}{h}a_{in+i}\Big(u(x_{n+in+i})-u(x)\Big) \\
&\quad+\frac{1}{h}\sum_{j=1}^{i-1}a_{in+j}\Big(u(x_{n+jn+i})-u(x)\Big)+\frac{1}{h}\sum_{j=i+1}^{n}a_{in+j}\Big(u(x_{n+in+j})-u(x)\Big) \\
&\quad+\frac{1}{h}\sum_{j=1}^{i-1}a_{ij}\Big(u(x_{jn+i})-u(x)\Big)+\frac{1}{h}\sum_{j=i+1}^{n}a_{ij}\Big(u(x_{jn+i})-u(x)\Big).
\end{aligned}$$

Therefore

$$
\begin{aligned}
\sum_{i=1}^{2n} f(x_i,x) &= \frac{1}{h}\sum_{i=1}^{n}\Big(\sum_{j=1}^{2n} a_{n+ij}\Big)\Big(u(x)-u(x_i)\Big) + \frac{1}{h}\sum_{i=1}^{n}\Big(\sum_{j=1}^{2n} a_{ij}\Big)\Big(u(x)-u(x_{n+i})\Big) \\
&+ \frac{1}{h}\sum_{i=1}^{n} a_{n+ii}\Big(u(x_{ii})-u(x)\Big) + \frac{1}{h}\sum_{i=1}^{n} a_{in+i}\Big(u(x_{n+in+i})-u(x)\Big) \\
&+ \frac{1}{h}\sum_{1\le j<i\le n} a_{n+ij}\Big(u(x_{ji})-u(x)\Big) + \frac{1}{h}\sum_{1\le i<j\le n} a_{n+ij}\Big(u(x_{ij})-u(x)\Big) \\
&+ \frac{1}{h}\sum_{1\le j<i\le n} a_{in+j}\Big(u(x_{n+jn+i})-u(x)\Big) + \frac{1}{h}\sum_{1\le i<j\le n} a_{in+j}\Big(u(x_{n+in+j})-u(x)\Big) \\
&+ \frac{1}{h}\sum_{1\le j<i\le n} a_{n+in+j}\Big(u(x_{in+j})-u(x)\Big) + \frac{1}{h}\sum_{1\le i<j\le n} a_{n+in+j}\Big(u(x_{in+j})-u(x)\Big) \\
&+ \frac{1}{h}\sum_{1\le j<i\le n} a_{ij}\Big(u(x_{jn+i})-u(x)\Big) + \frac{1}{h}\sum_{1\le i<j\le n} a_{ij}\Big(u(x_{jn+i})-u(x)\Big) \\
&= \frac{1}{h}\sum_{j=1}^{n}\Big(\sum_{i=1}^{2n} a_{n+ji}\Big)\Big(u(x)-u(x_j)\Big) + \frac{1}{h}\sum_{j=1}^{n}\Big(\sum_{i=1}^{2n} a_{ji}\Big)\Big(u(x)-u(x_{n+j})\Big) \\
&+ \frac{1}{h}\sum_{j=1}^{n} a_{n+jj}\Big(u(x_{jj})-u(x)\Big) + \frac{1}{h}\sum_{j=1}^{n} a_{jn+j}\Big(u(x_{n+jn+j})-u(x)\Big) \\
&+ \frac{1}{h}\sum_{1\le i<j\le n} (a_{n+ij}+a_{n+ji})\Big(u(x_{ij})-u(x)\Big) \\
&+ \frac{1}{h}\sum_{1\le i<j\le n} (a_{in+j}+a_{jn+i})\Big(u(x_{n+in+j})-u(x)\Big) \\
&+ \frac{1}{h}\sum_{1\le i<j\le n} (a_{ij}+a_{n+jn+i})\Big(u(x_{jn+i})-u(x)\Big) \\
&+ \frac{1}{h}\sum_{1\le i<j\le n} (a_{ji}+a_{n+in+j})\Big(u(x_{in+j})-u(x)\Big). \qquad \blacksquare
\end{aligned}
$$

From the expression of $\mathsf{div}\,(\mathsf{A}\nabla u)$ given in the above proposition we conclude that in general it is a second order difference operator on $\mathcal{C}(V_h)$. However, in some cases $\mathsf{div}\,(\mathsf{A}\nabla u)$ also can be a first order difference operator on $\mathcal{C}(V_h)$. In the following result we show the necessary and sufficient conditions in order for $\mathsf{div}\,(\mathsf{A}\nabla u)$ to be a first order difference operator.

Corollary 3.2 *Consider $\widehat{A}=(\widehat{a}_{ij})$ the $2n$-order matrix defined by*

$$\widehat{a}_{ij}=\widehat{a}_{n+jn+i}=\frac{1}{2}\,(a_{ij}+a_{n+jn+i}),\ \widehat{a}_{n+ij}=\frac{1}{2}\,(a_{n+ij}+a_{n+ji}),\ \widehat{a}_{in+j}=\frac{1}{2}\,(a_{in+j}+a_{jn+i}),\ i,j=1,\dots,n$$

and $\widehat{\mathsf{A}}$ the homogeneous field of matrices determined by it. Then for each $u\in\mathcal{C}(V_h)$ it is verified that $-\mathsf{div}\,(\mathsf{A}\nabla u)=-\mathsf{div}\,(\widehat{\mathsf{A}}\nabla u)$. In particular, $-\mathsf{div}\,(\mathsf{A}\nabla u)$ is a first order difference operator iff $\widehat{\mathsf{A}}$

is a diagonal field of matrices. Moreover $\widehat{\mathsf{A}}$ *is symmetric when* A *is a symmetric field and* $\widehat{\mathsf{A}}$ *is a metric tensor if, in addition,* A *is a metric tensor.*

Proof. Let $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u)$. Then, from the above proposition for each $x \in V_h$ we get that

$$\begin{aligned}
\mathcal{L}_h(u)(x) = & \frac{1}{h^2}\sum_{j=1}^{n}\Big(\sum_{i=1}^{2n}\widehat{a}_{n+ji}\Big)\Big(u(x)-u(x_j)\Big) + \frac{1}{h^2}\sum_{j=1}^{n}\Big(\sum_{i=1}^{2n}\widehat{a}_{ji}\Big)\Big(u(x)-u(x_{n+j})\Big)\\
- & \frac{1}{2h^2}\sum_{j=1}^{n}\widehat{a}_{n+jj}\Big(u(x)-u(x_{jj})\Big) - \frac{1}{2h^2}\sum_{j=1}^{n}\widehat{a}_{jn+j}\Big(u(x)-u(x_{n+jn+j})\Big)\\
- & \frac{1}{h^2}\sum_{1\le i<j\le n}\widehat{a}_{n+ij}\Big(u(x)-u(x_{ij})\Big) - \frac{1}{h^2}\sum_{1\le i<j\le n}\widehat{a}_{in+j}\Big(u(x)-u(x_{n+in+j})\Big)\\
- & \frac{1}{h^2}\sum_{\substack{i,j=1\\ i\ne j}}^{n}\widehat{a}_{ij}\Big(u(x)-u(x_{jn+i})\Big).
\end{aligned}$$

and therefore $\mathcal{L}_h(u)(x) = -\mathsf{div}\,(\widehat{\mathsf{A}}\nabla u)(x)$, tacking into account that $\widehat{a}_{in+j} = \widehat{a}_{jn+i}$ and $\widehat{a}_{n+ij} = \widehat{a}_{n+ji}$ for $1 \le i, j \le n$.

From the last expression we easily conclude that $\mathcal{L}_h(u)$ is a first order operator iff $\widehat{\mathsf{A}}$ is a diagonal matrix. On the other hand, $\widehat{A}$ is a symmetric matrix iff $\widehat{a}_{ij} = \widehat{a}_{n+in+j}$ and $\widehat{a}_{n+ij} = \widehat{a}_{in+j}$ for all $i, j = 1, \ldots, n$ and these identities are verified when A is a symmetric matrix, that is when A is a symmetric field. In this case, if for $x, y \in \mathbb{R}^n$ we consider $z = (x, y)^t$ and $w = (y, x)^t$, then we have that $z^t\widehat{A}z = \frac{1}{2}\,(z^tAz + w^tAw)$. Therefore if λ is the lowest eigenvalue of A the above identity implies that $z^t\widehat{A}z \ge \lambda\,|z|^2$, where $|z|$ is the euclidean length of z. ∎

We will deal with homogeneous discrete operators of the form

$$\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + qu,$$

where $q \in \mathbb{R}$, b is the homogeneous vector field determined by $b = (b_j) \in \mathbb{R}^{2n}$ and A is the homogeneous field of matrices determined by A. Moreover, from above corollary we can suppose without loss of generality that the coefficients of A verify the identities $a_{n+in+j} = a_{ji}$, $a_{in+j} = a_{jn+i}$ and $a_{n+ij} = a_{n+ji}$ for all $i, j = 1, \ldots, n$. Thus, in the sequel we assume these equalities and hence, for each $u \in \mathcal{C}(V_h)$ the values of function $\mathcal{L}_h(u)$ on V_h are given by

$$\begin{aligned}
\mathcal{L}_h(u)(x) = & \frac{1}{h^2}\sum_{j=1}^{n}(r_{n+j}-hb_j)\,\Big(u(x)-u(x_j)\Big) + \frac{1}{h^2}\sum_{j=1}^{n}(r_j-hb_{n+j})\,\Big(u(x)-u(x_{n+j})\Big)\\
- & \frac{1}{2h^2}\sum_{j=1}^{n}a_{n+jj}\Big(u(x)-u(x_{jj})\Big) - \frac{1}{2h^2}\sum_{j=1}^{n}a_{jn+j}\Big(u(x)-u(x_{n+jn+j})\Big)\\
- & \frac{1}{h^2}\sum_{1\le i<j\le n}a_{n+ij}\Big(u(x)-u(x_{ij})\Big) - \frac{1}{h^2}\sum_{1\le i<j\le n}a_{in+j}\Big(u(x)-u(x_{n+in+j})\Big)\\
- & \frac{1}{h^2}\sum_{\substack{i,j=1\\ i\ne j}}^{n}a_{ij}\Big(u(x)-u(x_{jn+i})\Big),
\end{aligned} \tag{14}$$

where $r_j = \sum\limits_{i=1}^{2n} a_{ji}$, $j = 1, \dots, 2n$ are the components of the homogeneous field r_A.

Up to now we have built a basic difference calculus on Γ_h for fixed $h > 0$. Of course, if a_{ij}, $i, j = 1, \dots, 2n$, b_j, $j = 1, \dots, 2n$ and q are functions of h, the above vectorial discrete calculus is in force for each $h > 0$. So, in the sequel we suppose that $q\colon (0, \infty) \longrightarrow \mathbb{R}$, the homogeneous flow b is determined by $b = (b_j)$, with $b_j\colon (0, +\infty) \longrightarrow \mathbb{R}$, $j = 1, \dots, 2n$, and the homogeneous field of matrices A is determined by $A = (a_{ij})$ with $a_{ij}\colon (0, +\infty) \longrightarrow \mathbb{R}$, $i, j = 1, \dots, 2n$.

We say that A is a field of Z-*matrices,* Z-*field* in short, if there exists $h_0 > 0$ such that for all $0 < h \le h_0$, $a_{ij}(h) \le 0$, $i, j = 1, \dots, 2n$, $i \ne j$. We say that A is a *diagonally dominant field of* M-*matrices, d.d.* M-*field* in short, if it is a Z-field and there exists $h_0 > 0$ such that for all $0 < h \le h_0$, $r_j(h) \ge 0$, $j = 1, \dots, 2n$. We say that A is a *strictly diagonally dominant field of* M-*matrices, s.d.d.* M-*field* in short if A is a d.d. M-field and there exists $C > 0$ such that $r_j(h) \ge C$, for all $j = 1, \dots, 2n$ and for h small enough. It is clear that if A is a Z-field, a d.d. M-field or a s.d.d. M-field, then $A(h)$ is a Z-matrix, a d.d. M-matrix or a s.d.d M-matrix respectively.

Comparing identity (14) with identity (2), we see that the difference operator $\mathcal{L}_h(u)$ is formally a difference scheme with constant coefficients. Our next goal is to prove that these discrete operators describe all difference schemes with constant coefficients.

Proposition 3.3 *If L_h is a difference scheme with constant coefficients then there exist a function q, homogeneous fields of matrices* A *and homogeneous vector fields* b *such that*

$$L_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + q\,u.$$

Moreover, b *can be chosen as a flow in which case* A *and* b *are uniquely determined.*

Proof. Suppose that the scheme L_h is given by

$$L_h(u)(x) = q\,u(x) + \sum_{j=1}^{2n} \gamma_j \Big(u(x) - u(x_j)\Big) + \sum_{\substack{1 \le i \le j \le 2n \\ j \ne n+i}} \gamma_{ij} \Big(u(x) - u(x_{ij})\Big), \qquad x \in V_h$$

and consider the difference operator $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + qu$ where b is the homogeneous vector field determined by $b = (b_j) \in \mathbb{R}^{2n}$ and A is the homogeneous field of matrices determined by A. From identity (14), we have that $\mathcal{L}_h(u) = L_h(u)$ for all $u \in \mathcal{C}(V_h)$ iff

$$a_{ij} = -h^2\gamma_{n+ij}, \quad a_{ji} = -h^2\gamma_{in+j}, \quad a_{in+j} = -h^2\gamma_{n+in+j}, \quad a_{n+ij} = -h^2\gamma_{ij}, \tag{15}$$

for all $1 \le i < j \le n$ and

$$a_{jn+j} = -2h^2\gamma_{n+jn+j}, \;\; a_{n+jj} = -2h^2\gamma_{jj}, \;\; \sum_{i=1}^{2n} a_{n+ji} - hb_j = h^2\gamma_j, \;\; \sum_{i=1}^{2n} a_{ji} - hb_{n+j} = h^2\gamma_{n+j} \tag{16}$$

for all $j = 1, \ldots, n$. This implies that all off diagonal coefficients of A are uniquely determined from the coefficients of the scheme L_h. Moreover, if for each $j = 1, \ldots, n$ we define $\tilde{b}_j = \frac{1}{2}(b_j + b_{n+j})$ and $\hat{b}_j = \frac{1}{2}(b_j - b_{n+j})$, then $b_j = \tilde{b}_j + \hat{b}_j$, $b_{n+j} = \tilde{b}_j - \hat{b}_j$ and the above identities imply that

$$a_{jj} - h\hat{b}_j = h\tilde{b}_j + h^2\Big(\gamma_j + 2\gamma_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n}(\gamma_{ij} + \gamma_{n+ij})\Big), \qquad j = 1, \ldots, n,$$

$$a_{jj} + h\hat{b}_j = h\tilde{b}_j + h^2\Big(\gamma_{n+j} + 2\gamma_{n+jn+j} + \sum_{\substack{i=1\\ i\neq j}}^{n}(\gamma_{in+j} + \gamma_{n+in+j})\Big), \quad j = 1, \ldots, n.$$

Solving this system we obtain that

$$a_{jj} = h\tilde{b}_j + \frac{h^2}{2}\Big(\gamma_j + \gamma_{n+j} + 2(\gamma_{jj} + \gamma_{n+jn+j}) + \sum_{\substack{i=1\\ i\neq j}}^{n}(\gamma_{ij} + \gamma_{n+ij} + \gamma_{in+j} + \gamma_{n+in+j})\Big), \quad j = 1, \ldots, n \tag{17}$$

and

$$\hat{b}_j = -\frac{h}{2}\Big(\gamma_j - \gamma_{n+j} + 2(\gamma_{jj} - \gamma_{n+jn+j}) + \sum_{\substack{i=1\\ i\neq j}}^{n}(\gamma_{ij} + \gamma_{n+ij} - \gamma_{in+j} - \gamma_{n+in+j})\Big), \quad j = 1, \ldots, n. \tag{18}$$

In addition, b is a flow iff $b_{n+j} = -b_j$, for all $j = 1, \ldots, n$, that is, iff $\tilde{b}_j = 0$, $j = 1, \ldots, n$, in which case the above equations determine uniquely both the diagonal coefficients of A and the coefficients of b. ∎

4 Symmetry, positivity and consistency

Thought this section we consider the difference operator $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + q\,u$, where b is the homogeneous flow determined by $b = (b_j)$ and A is the homogeneous field of matrices determined by $A = (a_{ij})$, whose coefficients verify the identities $a_{n+in+j} = a_{ji}$, $a_{in+j} = a_{jn+i}$ and $a_{n+ij} = a_{n+ji}$ for all $i, j = 1, \ldots, n$.

Our objective is to consider $\mathcal{L}_h$ as a difference scheme and then to analyze properties such as symmetry, positivity and consistency in terms of b and A. For this we will keep the notations of Proposition 3.3 which shows the relation between the homogeneous fields A and b, and the coefficients of the scheme. Specifically, from identities (15), (16) and (17) we have that

$$A = -h^2 \begin{bmatrix} -h^{-2}a_{11} & \gamma_{n+12} & \cdots & \gamma_{n+1n} & 2\gamma_{n+1n+1} & \gamma_{n+1n+2} & \cdots & \gamma_{n+12n} \\ \gamma_{n+21} & -h^{-2}a_{22} & \cdots & \gamma_{n+2n} & \gamma_{n+2n+1} & 2\gamma_{n+2n+2} & \cdots & \gamma_{n+22n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{2n1} & \gamma_{2n2} & \cdots & -h^{-2}a_{nn} & \gamma_{2nn+1} & \gamma_{2nn+2} & \cdots & 2\gamma_{2n2n} \\ 2\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} & -h^{-2}a_{11} & \gamma_{1n+2} & \cdots & \gamma_{12n} \\ \gamma_{21} & 2\gamma_{22} & \cdots & \gamma_{2n} & \gamma_{2n+1} & -h^{-2}a_{22} & \cdots & \gamma_{22n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & 2\gamma_{nn} & \gamma_{nn+1} & \gamma_{nn+2} & \cdots & -h^{-2}a_{nn} \end{bmatrix} \tag{19}$$

On the other hand, we know that $\mathcal{L}_h$ is a first order operator iff A is a diagonal field of matrices. In this case, if $\mathsf{c} = \mathsf{b} - \frac{1}{h}\mathsf{d}_{\mathsf{A}}$, then $\mathcal{L}_h(u) = \langle \mathsf{c}, \nabla u\rangle + q\,u$ and hence the study of second order difference operators of the form $-\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + q\,u$ where b is a homogeneous flow, includes the study of first order difference operators of the form $\langle \mathsf{c}, \nabla u\rangle + q\,u$, where c is an arbitrary homogeneous field. Moreover, we have that $\mathcal{L}_h$ is a first order difference operator iff it is associated with the difference scheme $L_h(u)(x) = q\,u(x) + \sum\limits_{j=1}^{2n} \gamma_j \Big(u(x) - u(x_j)\Big)$, where $\gamma_j = \dfrac{1}{h^2}\,(a_{jj} - hb_j)$ and $\gamma_{n+j} = \dfrac{1}{h^2}\,(a_{jj} + hb_j)$, $j = 1, \ldots, n$.

Proposition 4.1 *The following properties hold:*

i) *$\mathcal{L}_h$ is a quasi-symmetric scheme iff A is symmetric. Moreover, $\mathcal{L}_h$ is a symmetric scheme iff A is symmetric and in addition, $\mathsf{b} = 0$.*

ii) *$\mathcal{L}_h$ is a non negative scheme iff $q \geq 0$, A is a Z-field and $\mathsf{r}_{\mathsf{A}} \geq -h\mathsf{b}$ for h small enough. In particular, when $\mathcal{L}_h$ is a non negative scheme, then $r_j(h) + r_{n+j}(h) \geq 0$, $j = 1, \ldots, n$ and $\mathsf{d}_{\mathsf{A}} \geq h|\mathsf{b}|$, for h small enough.*

iii) *If $\lim\limits_{h\to 0} h\mathsf{b} = 0$, then $\mathcal{L}_h$ is of positive type iff $q \geq 0$ and A is a s.d.d.* M*-field.*

iv) *If $\mathsf{b} = 0$, then $\mathcal{L}_h$ is a scheme of non negative type iff $q \geq 0$ and A is a d.d.* M*-field.*

v) *If $\mathcal{L}_h$ is a quasi-symmetric scheme then it is of non negative type iff $q \geq 0$ and A is a symmetric d.d.* M*-field with $\mathsf{r}_{\mathsf{A}} \geq h|\mathsf{b}|$.*

Proof. (i) The scheme $\mathcal{L}_h$ is quasi-symmetric iff $a_{in+j} = a_{n+ij}$ and $a_{ij} = a_{ji}$, for all $i, j = 1, \ldots, n$, that is, iff A is symmetric. Moreover, in this case $r_j = r_{n+j}$ and hence from the two last equations of (16), we obtain that $h^2(\gamma_{n+j} - \gamma_j) = 2hb_j$. So, $\mathcal{L}_h$ is symmetric iff A is symmetric and $b_j = 0$, for all $j = 1, \ldots, n$.

(ii) It is straightforward from the nonnegativity definition and the previous relations.

(iii) If the scheme is of positive type, then A is a Z-field. Moreover, as $\lim_{h\to 0} hb_j = 0$, from identities $r_j + hb_j = \gamma_{n+j}$ and $r_{n+j} - hb_j = \gamma_j$ we get that there exists $C > 0$ such that $h^2\gamma_j \geq C$ for all $j = 1, \ldots, 2n$ and for all h small enough iff there exists $\hat{C} > 0$ such that $r_j(h) \geq \hat{C}$ for all $j = 1, \ldots, 2n$ and for all h small enough.

(iv) In this case $r_j = h^2\gamma_{n+j}$ and $r_{n+j} = h^2\gamma_j$, $j = 1, \ldots, n$, so the conclusion follows directly.

(v) If $\mathcal{L}_h$ is a quasi-symmetric scheme, from part (i) A is symmetric, which implies that $r_j = r_{n+j}$, $j = 1, \ldots, n$. Now, $\mathcal{L}_h$ is a scheme of non negative type iff $q \geq 0$, $a_{ij} \leq 0$ for all $i, j = 1, \ldots, 2n$ with $i \neq j$ and $r_j \geq h|b_j|$, $j = 1, \ldots, n$. ∎

Again the scheme $\mathcal{L}_h$ would be consistent if the coefficients of matrix A and vector b verified some conditions. On the other hand, when $K = 0$ or equivalently, when L is a first order differential operator, we will consider only first order difference operators or equivalently, A will be a diagonal field of matrices.

Next, we describe these conditions by using identities (3) to (6) and the expression of b and A in terms of the coefficients of the scheme given by identities (18) and (19), respectively.

When $m = 1$, equations (3) can be rewritten as

$$\phi^0 = k_0 - q, \qquad h\phi^1_j = k_j - 2b_j, \;\; j = 1, \ldots, n. \tag{20}$$

When $m = 2$, equations (4) can be rewritten as

$$\begin{aligned} h^2\phi^2_j =&\;\; 2a_{jj} - a_{jn+j} - a_{n+jj} - 2k_{jj}, && j = 1, \ldots, n, \\ h^2\psi^{11}_{ij} =&\;\; a_{ij} + a_{ji} - a_{in+j} - a_{n+ij} - 2k_{ij}, && 1 \leq i < j \leq n. \end{aligned} \tag{21}$$

When $m \geq 3$, equations (5) can be rewritten as

$$\begin{aligned} h^2\phi^k_j =&\;\; -2hb_j + (2^{k-1} - 1)\,(a_{jn+j} - a_{n+jj}), && k \text{ odd}, && j = 1, \ldots, n, \\ h^2\phi^k_j =&\;\; 2a_{jj} - (2^{k-1} - 1)\,(a_{jn+j} + a_{n+jj}), && k \text{ even}, && j = 1, \ldots, n, \end{aligned} \tag{22}$$

whereas equations (6) can be reformulated as

$$h^2\psi^{lk-l}_{ij} = \begin{cases} -a_{ij} - a_{ji} - a_{in+j} - a_{n+ij}, & \text{if } k = 2r,\, l = 2s,\, s = 1, \ldots, r-1, \\ \;\; a_{ij} + a_{ji} - a_{in+j} - a_{n+ij}, & \text{if } k = 2r,\, l = 2s-1,\, s = 1, \ldots, r, \\ \;\; a_{ij} - a_{ji} + a_{in+j} - a_{n+ij}, & \text{if } k = 2r-1,\, l = 1, \ldots, r-1, \\ -a_{ij} + a_{ji} + a_{in+j} - a_{n+ij}, & \text{if } k = 2r-1,\, l = r, \ldots, 2(r-1), \end{cases} \tag{23}$$

for each $k = 3, \ldots, m$, $l = 1, \ldots, k-1$ and $1 \leq i < j \leq n$. Therefore, Proposition 2.1 becomes in the following result.

Proposition 4.2 *Consider $m \geq 1$, suppose that $K = 0$ when $m = 1$ and that the following conditions are verified*

$$a_{ij},\ hb_k \in O(h^s), \quad k = 1, \dots, n, \quad i, j = 1, \dots, 2n,$$

$$\phi^0 \in O(h^r),\ \phi_j^k \in O(h^{r-k}), \quad k = 1, \dots, m, \quad j = 1, \dots, n,$$

$$\psi_{ij}^{lk-l} \in O(h^{r-k}), \quad k = 2, \dots, m, \quad l = 1, \dots, k-1, \quad 1 \leq i < j \leq n,$$

where $r > 0$ and $s > 1 - m$. Then, the scheme $\mathcal{L}_h$ is consistent of order $\min\{r, s + m - 1\}$.

Identities (20) to (23) form a system with $1 + 2n(n+1)$ unknowns and $1 + mn + \binom{n}{2}\binom{m}{2}$ equations, in which the functions ϕ^0, $\{\phi_j^k\}$ and $\{\psi_{ij}^{lk-l}\}$ are the data. Moreover, when $K = 0$ we add the condition A is diagonal, that is the identities $a_{ij} = 0$ when $i \neq j$.

So, when $K \neq 0$, if $1 \leq m \leq 3$ the system is compatible and indeterminate whereas if $m \geq 4$, it will be necessary to add compatibility conditions. On the other hand, when $K = 0$, if $m = 1$ the system is compatible and indeterminate whereas if $m \geq 2$ it will be necessary to add compatibility conditions.

If $m = 1$, the solutions of the system are given by

$$q = k_0 - \phi^0, \quad b_j = \frac{k_j}{2} - \frac{h}{2}\phi_j^1, \quad j = 1, \dots, n \tag{24}$$

and therefore $0 < s \leq 1$, since in this case $\mathsf{k} \neq \mathsf{0}$.

If $m = 2$, the solutions of the system are given by

$$\left.\begin{aligned}
q &= k_0 - \phi^0, \qquad b_j = \frac{k_j}{2} - \frac{h}{2}\phi_j^1, && j = 1, \dots, n,\\
a_{jj} &= \frac{1}{2}\left(a_{jn+j} + a_{n+jj}\right) + k_{jj} + \frac{h^2}{2}\,\phi_j^2, && j = 1, \dots, n,\\
a_{ji} &= a_{in+j} + a_{n+ij} - a_{ij} + 2k_{ji} + h^2\psi_{ij}^{11}, && 1 \leq i < j \leq n.
\end{aligned}\right\} \tag{25}$$

Therefore, when $K = 0$ the system is compatible only when $\psi_{ij}^{11} = 0$, $1 \leq i < j \leq n$, since in this case A must be diagonal, and in this case $a_{jj} = \dfrac{h^2}{2}\,\phi_j^2 \in O(h^r)$, $j = 1, \dots, n$, which implies that $s = \min\{1, r\}$. On the other hand, when $K \neq 0$ necessarily $-1 < s \leq 0$ and the system is compatible and indeterminate.

If $m = 3$, the solutions of the system are given by

$$\left.\begin{aligned}
q &= k_0 - \phi^0, \qquad b_j = \frac{k_j}{2} - \frac{h}{2}\phi_j^1, && j = 1, \dots, n,\\
a_{jj} &= k_{jj} + \frac{h}{6}\,k_j + a_{n+jj} - \frac{h^2}{6}\left(\phi_j^1 - 3\phi_j^2 - \phi_j^3\right), && j = 1, \dots, n,\\
a_{jn+j} &= \frac{h}{3}\,k_j + a_{n+jj} - \frac{h^2}{3}\left(\phi_j^1 - \phi_j^3\right) && j = 1, \dots, n
\end{aligned}\right\} \tag{26}$$

and by

$$\left.\begin{aligned} a_{ij} &= k_{ij} + a_{n+ij} + \frac{h^2}{2}\,(\psi_{ij}^{11} + \psi_{ij}^{12}), \quad 1 \le i < j \le n \\ a_{ji} &= k_{ij} + a_{n+ij} + \frac{h^2}{2}\,(\psi_{ij}^{11} + \psi_{ij}^{21}), \quad 1 \le i < j \le n \\ a_{in+j} &= a_{n+ij} + \frac{h^2}{2}\,(\psi_{ij}^{12} + \psi_{ij}^{21}), \qquad\quad 1 \le i < j \le n. \end{aligned}\right\} \tag{27}$$

When $K = 0$ these conditions imply that the system is compatible only when $\psi_{ij}^{11} = \psi_{ij}^{12} = \psi_{ij}^{21} = 0$, $1 \le i < j \le n$ and $\phi_j^3 = \dfrac{1}{h}\,(h\phi_j^1 - k_j)$, $j = 1, \ldots, n$ and hence $r = 2$. In this case $a_{jj} = \dfrac{h^2}{2}\,\phi_j^2 \in O(h^2)$ and $hb_j \in O(h)$, $j = 1, \ldots, n$, which implies that $s = 1$. On the other hand, when $K \ne 0$ necessarily $-2 < s \le \min\{0, r-1\}$ and the system is compatible and indeterminate.

Definitely, when $1 \le m \le 3$ the coefficients of the field of matrices A and the flow b depend on the choice of ϕ^0, $\{\phi_j^1, \phi_j^2, \phi_j^3\}_{j=1}^n$ and $\{\psi_{ij}^{11}, \psi_{ij}^{12}, \psi_{ij}^{21}\}_{i<j}$ (and at least on the parameters $\{a_{n+ij}\}_{i\le j}$.) In any case consistency implies that $\mathsf{b} = \mathsf{0}$ iff $\mathsf{k} = \mathsf{0}$ and $\phi_1^1 = \cdots = \phi_n^1 = 0$ and also that $\mathsf{b} \in O(1)$ and hence $\lim\limits_{h\to 0} h\mathsf{b} = \mathsf{0}$.

In the following propositions we summarize the above results and study what difference schemes are consistent with first or second order differential operators. In what follows we denote by $\hat{\mathsf{k}}$ and by $\hat{\phi}^1$ the homogeneous flows determined by k and by $\phi^1 = h(\phi_1^1, \ldots, \phi_n^1)$, respectively.

Proposition 4.3 *Suppose that $K = 0$, that is, $L(u) = \langle \mathsf{k}, \nabla u\rangle + k_0 u$, where $\mathsf{k} \ne \mathsf{0}$ and consider the difference operator $\mathcal{L}_h(u) = \langle \mathsf{c}, \nabla u\rangle + qu$, where c is an homogeneous vector field and $q \in \mathcal{C}(V_h)$. Then $\mathcal{L}_h$ is a consistent scheme iff there exist $\phi^0, \phi^1 \in O(h^r)$, $r > 0$ and $\mathsf{a} \in O(h^{s-1})$, $s > 0$, the homogeneous field determined by $a \in \mathbb{R}^{2n}$ where $a_j = a_{n+j}$, $j = 1, \ldots, n$, such that $q = k_0 - \phi^0$ and $\mathsf{c} = \mathsf{a} + \frac{1}{2}\,(\hat{\mathsf{k}} - \hat{\phi}^1)$. Moreover, the order of consistency is $\min\{r, s, 2\}$ and 2 is the greatest order of consistency.*

Proposition 4.4 *Suppose that $L(u) = -\mathsf{div}\,(K\nabla u) + \langle \mathsf{k}, \nabla u\rangle + k_0 u$ with $K \ne 0$ and consider the difference operator $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + qu$, where A is a field of matrices, b is a flow and $q \in \mathcal{C}(V_h)$. Then $\mathcal{L}_h$ is a consistent scheme iff one of the following properties holds:*

i) There exist $M_1, M_2, M_3 \in O(h^s)$, $s \in (-1, 0]$, $\phi^0, \phi^1, \Phi \in O(h^r)$, $r > 0$, where M_1, M_2, Φ are symmetric and M_3 is skew-symmetric, such that $q = k_0 - \phi^0$, $\mathsf{b} = \frac{1}{2}\,(\hat{\mathsf{k}} - \hat{\phi}^1)$ and

$$A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\,(M_1 + M_2 + M_3) & M_1 \\ M_2 & \frac{1}{2}\,(M_1 + M_2 - M_3) \end{bmatrix} + \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \tag{28}$$

Moreover, the order of consistency is $\min\{r, s+1\} \le 1$, at least.

ii) There exist $M \in O(h^s)$, $s \in (-2, \min\{0, r-1\}]$, $\phi^0, \phi^1, \Phi, h\Psi \in O(h^r)$, $r > 0$, where M, Φ are symmetric such that $q = k_0 - \phi^0$, $\mathsf{b} = \frac{1}{2}(\hat{\mathsf{k}} - \hat{\phi}^1)$ and

$$A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} M & M \\ M & M \end{bmatrix} - \begin{bmatrix} D & 2D \\ 0 & D \end{bmatrix} + \begin{bmatrix} \Phi + \Psi & \Psi + \Psi^t \\ 0 & \Phi + \Psi^t \end{bmatrix} \tag{29}$$

where D is the diagonal matrix whose non null entries are $\frac{h}{6}(h\phi_j^1 - k_j)$, $j = 1, \ldots, n$. Moreover, the order of consistency is $\min\{r, s+2\} \leq 2$, at least.

In any case, L is a selfadjoint differential operator iff $\lim\limits_{h \to 0} \mathsf{b} = 0$.

Corollary 4.5 *A second order differential operator $L(u) = -\mathsf{div}\,(K\nabla u) + \langle \mathsf{k}, \nabla u\rangle + k_0 u$, has consistent schemes that are first order difference operators iff K is a diagonal matrix. Moreover $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + qu$ is one of them iff $q = k_0 - \phi^0$, $\mathsf{b} = \frac{1}{2}(\hat{\mathsf{k}} - \hat{\phi}^1)$ and*

$$A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix},$$

where $\phi^0, \phi^1, \Phi \in O(h^r)$, $r > 0$ and Φ is a diagonal matrix. In addition, if $\mathsf{c} = -\frac{1}{h}\mathsf{d}_{\mathsf{A}} + \frac{1}{2}(\hat{\mathsf{k}} - \hat{\phi}^1)$, then $\mathcal{L}_h(u) = \langle \mathsf{c}, \nabla u\rangle + qu$ and its order of consistency is $\min\{r, 2\}$.

If we take $r \geq 2$ and $s = 0$, Proposition 4.4 (ii) assures the existence of difference operators that are 2-consistent schemes. However, unlike the first order differential operator case, the above proposition does not exclude that difference operators of greater order of consistency could be built. Of course, to analyze this situation we must consider $m = 4$ and $r > 2$ and moreover, we must add to the identities (26) and (27) the equations (22) and (23) corresponding to the derivatives of order 4:

$$\left.\begin{aligned} h^2\phi_j^4 &= 2a_{jj} - 7(a_{jn+j} + a_{n+jj}), && j = 1, \ldots, n, \\ h^2\psi_{ij}^{13} = h^2\psi_{ij}^{31} &= a_{ij} + a_{ji} - a_{in+j} - a_{n+ij}, && 1 \leq i < j \leq n, \\ -h^2\psi_{ij}^{22} &= a_{ij} + a_{ji} + a_{in+j} + a_{n+ij}, && 1 \leq i < j \leq n, \end{aligned}\right\} \tag{30}$$

where $\phi_j^4, \psi_{ij}^{13}, \psi_{ij}^{22} \in O(h^{r-4})$ for each $1 \leq i < j \leq n$. Then, the second equation of (21) and the second equation of (30) imply that $k_{ij} = h^2\psi_{ij}^{13} - h^2\psi_{ij}^{11} \in O(h^{r-2})$ and hence the system only can be compatible if $k_{ij} = 0$ for each $i, j = 1, \ldots, n$, $i \neq j$ and then necessarily $\psi_{ij}^{13} = \psi_{ij}^{11}$. Assuming this condition we get that the solution of the system is given by

$$\left.\begin{aligned} &q = k_0 - \phi^0, \quad b_j = \frac{k_j}{2} - \frac{h}{2}\phi_j^1, \\ &a_{jj} = \frac{7}{6}k_{jj} + \frac{h^2}{12}(7\phi_j^2 - \phi_j^4), \quad a_{jn+j} = \frac{k_{jj}}{6} + \frac{h}{6}k_j - \frac{h^2}{12}\Big(2\phi_j^1 - \phi_j^2 - 2\phi_j^3 + \phi_j^4\Big), \\ &a_{n+jj} = \frac{k_{jj}}{6} - \frac{h}{6}k_j + \frac{h^2}{12}\Big(2\phi_j^1 + \phi_j^2 - 2\phi_j^3 - \phi_j^4\Big), \end{aligned}\right\} \tag{31}$$

for all $j = 1, \ldots, n$, and by

$$\left.\begin{aligned} a_{ij} &= \frac{h^2}{4}\Big(\psi_{ij}^{11} + \psi_{ij}^{12} - \psi_{ij}^{21} - \psi_{ij}^{22}\Big), & a_{ji} &= \frac{h^2}{4}\Big(\psi_{ij}^{11} - \psi_{ij}^{12} + \psi_{ij}^{21} - \psi_{ij}^{22}\Big), \\ a_{in+j} &= -\frac{h^2}{4}\Big(\psi_{ij}^{11} - \psi_{ij}^{12} - \psi_{ij}^{21} + \psi_{ij}^{22}\Big), & a_{n+ij} &= -\frac{h^2}{4}\Big(\psi_{ij}^{11} + \psi_{ij}^{12} + \psi_{ij}^{21} + \psi_{ij}^{22}\Big), \end{aligned}\right\} \quad (32)$$

for all $1 \le i < j \le n$. In this case, the coefficients of the field of matrices A and the flow b are uniquely determined by the data ϕ^0, $\{\phi_j^k\}_{j=1}^n$, $k = 1, \ldots, 4$ and $\{\psi_{ij}^{11}, \psi_{ij}^{12}, \psi_{ij}^{21}, \psi_{ij}^{22}\}_{i<j}$.

Proposition 4.6 *Suppose that K, the matrix of leading coefficients of L is diagonal and non null. Consider $q = k_0 - \phi^0$, $\mathsf{b} = \frac{1}{2}\,(\hat{\mathsf{k}} - \hat{\phi}^1)$ and A the field of matrices determined by*

$$A = \frac{1}{6}\begin{bmatrix} 7K & K \\ K & 7K \end{bmatrix} - \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix} - \begin{bmatrix} \hat{D} - \Phi - \Psi + \Psi^t + \Theta & \hat{D} + \Phi - \Psi - \Psi^t + \Theta \\ \hat{D} + \Phi + \Psi + \Psi^t + \Theta & \hat{D} - \Phi + \Psi - \Psi^t + \Theta \end{bmatrix} \quad (33)$$

where $\phi^0, \phi^1, \Phi, h\Psi, h^2\Theta \in O(h^r)$, $r > 2$, Φ, Θ are symmetric matrices and D and $\hat{D}$ are the diagonal matrices whose non null entries are $\frac{h}{6}\,(h\phi_j^1 - k_j)$ and $-\frac{4}{3}\,\Phi_{jj}$, $j = 1, \ldots, n$ respectively. Then, the difference operator $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + qu$ is a consistent scheme of order $\min\{r, 4\}$ *and* 4 *is the greatest order of consistency. In particular, $\mathcal{L}_h$ is always a second order difference operator.*

Proof. As $r > 2$, by applying identities (31) and (32), necessarily $s = 0$. Therefore, from Proposition 4.2, the order of consistency of $\mathcal{L}_h$ is $\min\{r, 3\}$ at least.

To know if $\mathcal{L}_h$ has a more greater order of consistency we must consider $r > 3$, take $m = 5$ and add to identities (31) and (32) the following equations

$$\left.\begin{aligned} h^2\phi_j^5 &= -2hb_j + 15(a_{jn+j} - a_{n+jj}), && j = 1, \ldots, n, \\ \psi_{ij}^{14} = \psi_{ij}^{23} &= \psi_{ij}^{12}, \quad \psi_{ij}^{41} = \psi_{ij}^{32} = \psi_{ij}^{21}, && 1 \le i < j \le n, \end{aligned}\right\} \quad (34)$$

that correspond to the derivatives of order 5. Therefore $h^5\psi_{ij}^{14}, h^5\psi_{ij}^{41}, h^5\psi_{ij}^{23}, h^5\psi_{ij}^{32} \in O(h^{r+2})$ and replacing the value of the coefficients given in (31) in the first equation of (34), we obtain that ϕ_j^5 is uniquely determined by the expression

$$h\phi_j^5 = 4k_j - 4h\phi_j^1 + 5h\phi_j^3,\ j = 1, \ldots, n,$$

which implies that $h^5\phi_j^5 \in O(h^4)$ since $r > 3$. In conclusion, $\mathcal{L}_h$ has order of consistence $\min\{r, 4\}$ at least.

Suppose now that $r > 4$. We must take $m = 6$ to know whether or not $\mathcal{L}_h$ has order of consistency greater than 4. From (22), the new equations that we have to add are

$$h^2\phi_j^6 = 2a_{jj} - 31(a_{jn+j} + a_{n+jj}),\ j = 1, \ldots, n$$

where $\phi_j^6 \in O(h^{r-6})$ to assure consistency of order r. Replacing the value of the coefficients of the scheme given by (31) in the above equation we obtain that

$$h^2\phi_j^6 = -8k_{jj} - 4h^2\phi_j^2 + 5h^2\phi_j^4,\ j = 1, \ldots, n,$$

which imply that necessarily $k_{jj} = 0$ for all $j = 1, \ldots, n$.

Finally, if $\mathcal{L}_h$ was a first order difference operator, necessarily $a_{n+jj} = 0$ and hence from equation (31), $k_j = 0$, $j = 1, \cdots, n$. ∎

We have shown that any differential operator with constant coefficients admits 2-consistent difference schemes with constant coefficients, but only those second order differential operators which leading coefficient matrix is diagonal have schemes of higher order of consistency. Our next objective is to characterize the differential operators with constant coefficients that admit consistence difference schemes which are also symmetric or positive. In addition, we wish to obtain all difference schemes with the above properties.

From Proposition 4.1 it is clear that each difference scheme consistent with a first order differential operator is in fact a quasi-symmetric scheme, since any diagonal field of matrices is block-isotropic. So, all difference operators verifying the conditions of Proposition 4.3 are quasi-symmetric consistent schemes. In addition, they can not be symmetric. Next we summarize these results.

Proposition 4.7 *Each first order difference operator that is consistent with a first order differential operator is a quasi-symmetric but not symmetric scheme and* 2 *is the greatest order of consistency.*

Now, we look for quasi-symmetric consistent schemes for second order differential operators. As A needs to be a symmetric matrix to ensure quasi-symmetry, expression (28) must be rewritten as

$$A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} M & M \\ M & M \end{bmatrix} + \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}, \tag{35}$$

where $M = M_1 = M_2$.

On the other hand, imposing the quasi-symmetry condition to the expression (29) we obtain that $\Psi = D$ and hence (29) is rewritten as (35). Moreover, if $\mathsf{k} \neq 0$, necessarily $r = 2$.

Finally, when K is a non null diagonal matrix, imposing the quasi-symmetry condition to the expression (33) we get that $D = 2\Psi$ and hence $\mathsf{k} = 0$ since $r > 2$. Moreover, expression (33) is rewritten as

$$A = \frac{1}{6}\begin{bmatrix} 7K & K \\ K & 7K \end{bmatrix} - \begin{bmatrix} \hat{D} - \Phi + \Theta & \hat{D} + \Phi + \Theta \\ \hat{D} + \Phi + \Theta & \hat{D} - \Phi + \Theta \end{bmatrix}. \tag{36}$$

The following result is obtained straightforwardly from Proposition 4.4 by imposing the above condition.

Proposition 4.8 *Suppose that $K \neq 0$ and consider $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + q\,u$, where $q = k_0 - \phi^0$, $\mathsf{b} = \dfrac{1}{2}\,(\hat{\mathsf{k}} - \hat{\phi}^1)$ and A is the field of matrices determined by*

$$A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} M & M \\ M & M \end{bmatrix} + \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}$$

and where $\phi^0, \phi^1, \Phi \in O(h^r), r > 0$, $M \in O(h^s), s \in (-2, 0]$ and Φ and M are symmetric matrices. Then, $\mathcal{L}_h$ is a quasi-symmetric consistent scheme of order $\min\{r, s+2\}$ *and when $\mathsf{k} \neq 0$, the greatest order of consistency for this type of schemes is 2. Moreover, L has symmetric consistent schemes iff it is selfadjoint. In this case, if K is a diagonal matrix and matrix A verifies expression (36), then $\mathcal{L}_h$ is a symmetric consistent scheme of order* $\min\{r, 4\}$.

Next, we analyze a very special case of Propositions 4.7 and 4.8 in which the field of matrices A is isotropic, that is there exist a_1, a_2, a_3 such that $a_{jj} = a_1$, $a_{jn+j} = a_{n+jj} = a_2$, $j = 1, \ldots, n$, and $a_{ij} = a_3$, otherwise. This kind of difference schemes were studied in [2].

Proposition 4.9 *The differential operator L has consistent schemes such that A is an isotropic field of matrices iff $L(u) = -a\Delta u + \langle \mathsf{k}, \nabla u\rangle + k_0 u$, with $a \in \mathbb{R}$. Moreover, the difference operator $\mathcal{L}_h$ is one of them iff there exist $\phi \in O(h^r)$ with $r > 0$ and $a_2, a_3 \in O(h^s)$ with $s \in (-2, 0]$, such that*

$$A = (a + \phi) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + a_2 \begin{bmatrix} I & I \\ I & I \end{bmatrix} + a_3 \begin{bmatrix} J & J \\ J & J \end{bmatrix},$$

where I is the identity matrix and J is the matrix whose entries are all equal to 1. In addition, $a_2 = a_3 = 0$ when $a = 0$.

Proof. If L is a first order differential operator, that is, $L(u) = -a\Delta u + \langle \mathsf{k}, \nabla u\rangle + k_0 u$ with $a = 0$, then from Proposition 4.3 all consistent difference schemes are given by the choice of a diagonal field $\mathsf{A} \in O(h^s)$ with $s > 0$. So, A is isotropic iff $a_{jj} = a_1 \in O(h^s)$, for all $j = 1, \ldots, n$.

Suppose that L is a second order differential operator and that $\mathcal{L}_h$ is a consistent scheme where A is an isotropic field of matrices. Hence as A is a symmetric matrix, expression (35) must be verified, which imply that $(a_1 - a_2)I - K \in O(h^r)$. Taking limit when $h \to 0$ we obtain that $K = aI$ where $a = \lim\limits_{h\to 0}(a_1 - a_2)$. The converse can be deduced from the above results keeping in mind equations (35). ∎

Now we will deal with consistent schemes of non negative or positive type. From Proposition 4.1 (ii), $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + qu$ is of non negative type iff $q \geq 0$, A is a Z-field and $\mathsf{r}_\mathsf{A} \geq -h\mathsf{b}$ and hence $\mathsf{d}_\mathsf{A} \geq h|\mathsf{b}|$. In particular, when $\mathcal{L}_h$ is a first order difference operator then $\mathcal{L}_h(u) = \langle \mathsf{c}, \nabla u\rangle + qu$ where $\mathsf{c} = \mathsf{b} - \frac{1}{h}\,\mathsf{d}_\mathsf{A}$ and the last condition is equivalent to $\mathsf{c} \leq \mathsf{0}$. Moreover, consistency implies that at least equalities (24) or (25) must be verified and hence

$k_0 = \lim_{h\to 0} q \geq 0$. So, non negativity and consistency imply that $k_0 \geq 0$, $a_{ij} \leq 0$, $i, j = 1, \ldots, 2n$, $i \neq j$ and $\mathsf{r_A} \geq \frac{h}{2}\,(\hat\phi^1 - \hat{\mathsf{k}})$. In particular, it is also verified that $\mathsf{d_A} \geq \frac{h}{2}\,|\hat\phi^1 - \hat{\mathsf{k}}|$.

Proposition 4.10 *If L is a first order differential operator, then L has consistent and non negative type schemes iff $k_0 \geq 0$. In this case, the maximum order of consistency is 1 and none of them is of positive type. Moreover, if c is an homogeneous vector field and $q \in \mathcal{C}(V_h)$, then $\mathcal{L}_h(u) = \langle \mathsf{c}, \nabla u\rangle + qu$ is a consistent scheme of non negative type iff there exist $\phi^0, \phi^1 \in O(h^r)$, $r > 0$, and $\mathsf{a} \in O(h^{s-1})$, $s \in (0, 1]$, the homogeneous field determined by $a \in \mathbb{R}^{2n}$, verifying that $\phi^0 \leq 0$ when $k_0 = 0$, $a_{jj} = a_{n+jn+j}$, $j = 1, \ldots, n$, $q = k_0 - \phi^0$, $-\mathsf{a} \geq \frac{1}{2}\,|\hat\phi^1 - \hat{\mathsf{k}}|$ and $\mathsf{c} = \mathsf{a} + \frac{1}{2}\,(\hat{\mathsf{k}} - \hat\phi^1)$. In addition, the order of consistency of $\mathcal{L}_h$ is $\min\{r, s\}$.*

Proof. If L has consistent schemes of non negative type, necessarily $k_0 \geq 0$. Conversely, if $k_0 > 0$, then for any $\phi^0 \in O(h^r)$, we have that $q = k_0 - \phi^0 \geq 0$ for h small enough, whereas if $k_0 = 0$ then $q \geq 0$ iff $\phi^0 \leq 0$.

From Proposition 4.3, the difference operator $\mathcal{L}_h(u) = \langle \mathsf{c}, \nabla u\rangle + qu$, where c is an homogeneous vector field and $q \in \mathcal{C}(V_h)$ is a consistent scheme iff there exist $\phi^0, \phi^1 \in O(h^r)$, $r > 0$ and $\mathsf{a} \in O(h^{s-1})$, $s > 0$, the homogeneous field determined by $a \in \mathbb{R}^{2n}$ where $a_{jj} = a_{n+jn+j}$, $j = 1, \ldots, n$, such that $q = k_0 - \phi^0$ and $\mathsf{c} = \mathsf{a} + \frac{1}{2}\,(\hat{\mathsf{k}} - \hat\phi^1)$. Moreover, $\mathcal{L}_h$ is of non negative type iff $\mathsf{c} \leq 0$, that is, iff $-\mathsf{a} \geq \frac{1}{2}\,|\hat\phi^1 - \hat{\mathsf{k}}|$, which implies that $s \leq 1$ and hence the order of consistency is $\min\{r, s, 2\} = \min\{r, s\} \leq 1$.

On the other hand, as $\lim_{h\to 0} h^2\mathsf{c}(h) = 0$, $\mathcal{L}_h$ can not be of positive type. ∎

The following result, related with non negative schemes for second orden differential operators, is well known.

Proposition 4.11 *Second order differential operators do not have consistent schemes of non negative type of order greater than 2.*

Proof. We know that if a second order differential operator has a r-consistent scheme with $r > 2$, necessarily its leading coefficient matrix, K, must be diagonal. Therefore, suppose that this is the case and that $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + qu$ is a non negative consistent scheme with order of consistency greater than 2. Then, by applying Proposition 4.6, A is determined by

$$A = \frac{1}{6}\begin{bmatrix} 7K & K \\ K & 7K \end{bmatrix} - \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix} - \begin{bmatrix} \hat D - \Phi - \Psi + \Psi^t + \Theta & \hat D + \Phi - \Psi - \Psi^t + \Theta \\ \hat D + \Phi + \Psi + \Psi^t + \Theta & \hat D - \Phi + \Psi - \Psi^t + \Theta \end{bmatrix}$$

where $\phi^1, \Phi, h\Psi, h^2\Theta \in O(h^r)$, $r > 2$, Φ, Θ are symmetric matrices and D and $\hat D$ are the diagonal matrices whose non null entries are $\frac{h}{6}\,(h\phi^1_j - k_j)$ and $-\frac{4}{3}\,\Phi_{jj}$, $j = 1, \ldots, n$ respectively. Therefore,

$\lim_{h\to 0} A(h) = \frac{1}{6}\begin{bmatrix} 7K & K \\ K & 7K \end{bmatrix}$. On the other hand, A must be a Z-matrix with non negative diagonal coefficients which implies $K = 0$. ∎

At the sight of the above result, when L is a second order differential operator we can suppose that the difference operator

$$\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \langle \mathsf{b}, \nabla u\rangle + q\,u$$

determines a consistent scheme with order of consistency lower or equal to 2 and therefore the function q, the vector b and the coefficients of matrix A must satisfy the conditions of Proposition 4.4 (i), at least. The following proposition characterize those differential operators that have consistent difference schemes of non negative type. For $n = 2$, this results was obtained by D. Greenspan and P.C. Jain, [5].

Proposition 4.12 *If the operator L has consistent schemes of non negative type, then $k_0 \geq 0$ and its matrix of leading coefficients K is positive semidefinite and diagonally dominant, that is*

$$k_0 \geq 0 \quad \textit{and} \quad k_{jj} \geq \sum_{\substack{i=1\\ i\neq j}}^{n} |k_{ji}|, \qquad j = 1, \ldots, n.$$

Moreover, the converse holds when L is self-adjoint.

Proof. If L admits a consistent scheme of non negative type, necessarily $k_0 \geq 0$. Moreover, there exist $M_1, M_2, M_3 \in O(h^s)$, $s \in (-1, 0]$, $\phi^0, \phi^1, \Phi \in O(h^r)$, $r > 0$, where M_1, M_2, Φ are symmetric and M_3 is skew-symmetric, such that $q = k_0 - \phi^0$, $\mathsf{b} = \frac{1}{2}\,(\hat{\mathsf{k}} - \hat{\phi}^1)$ and

$$A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\,(M_1 + M_2 + M_3) & M_1 \\ M_2 & \frac{1}{2}\,(M_1 + M_2 - M_3) \end{bmatrix} + \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}.$$

From Proposition 4.1 (ii), as $\mathcal{L}_h$ is of non negative type, then $q \geq 0$, A is a Z-field, $\mathsf{r}_{\mathsf{A}} \geq -h\mathsf{b}$ and hence $M_1, M_2 \leq 0$. Moreover from the above equality we have that

$$\begin{aligned} a_{jj} = &\quad k_{jj} + \Phi_{jj} + \tfrac{1}{2}\,(a_{jn+j} + a_{n+jj}), \quad j = 1, \ldots, n, \\ a_{ij} + a_{ji} = &\quad 2(k_{ij} + \Phi_{ij}) + a_{in+j} + a_{n+ij}, \qquad 1 \leq i < j \leq n, \end{aligned}$$

which implies that

$$k_{jj} + \Phi_{jj} \geq a_{jj}, \;\; j = 1, \ldots, n, \qquad 2|k_{ij} + \Phi_{ij}| \leq -(a_{ij} + a_{ji} + a_{in+j} + a_{n+ij}), \;\; 1 \leq i < j \leq n.$$

Therefore,

$$2\sum_{\substack{i=1\\ i\neq j}}^{n} |k_{ij}+\Phi_{ij}| \leq a_{jj}+a_{jn+j}-r_j+a_{jj}+a_{n+jj}-r_{n+j} \leq 2a_{jj}-(r_j+r_{n+j}) \leq 2\,(k_{jj}+\Phi_{jj}),\; j = 1, \ldots, n.$$

The result follows by taking limits in the above expression.

To prove the converse when L is self-adjoint, it is suffices to choose $\hat{\phi}^1 = \mathsf{0}$, $M_3 = \Phi = 0$ and $M_1 = M_2 = -K^+$, where K^+ stands for the matrix with zero diagonal entries and whose off diagonal entries are given by $\frac{1}{2}(k_{ij} + |k_{ij}|)$, $i \neq j$. Then A is a d.d. M-field and the conclusion follows from Propositions 4.1 (iv) and 4.4 (i). ∎

Under the hypotheses of the above proposition, if $\mathcal{L}_h$ is a consistent scheme of non negative type then conditions $k_{jj} + \Phi_{jj} \geq a_{jj} \geq 0$ and $r_j + hb_j$, $r_{n+j} - hb_j \geq 0$, $j = 1, \ldots, n$ imply that

$$k_{jj} + \Phi_{jj} + hb_j \geq -a_{jn+i} \geq 0, \qquad k_{jj} + \Phi_{jj} - hb_j \geq -a_{n+ji} \geq 0, \qquad i, j = 1, \ldots, n,$$

$$k_{jj} + \Phi_{jj} + hb_j \geq -a_{ji} \geq 0, \qquad k_{jj} + \Phi_{jj} - hb_j \geq -a_{ij} \geq 0, \qquad 1 \leq i < j \leq n,$$

and hence $A \in O(1)$. In addition, when L is a self-adjoint differential operator with positive semidefinite and d.d. leading coefficients matrix, the difference scheme built in the proof of the above proposition, that is $-\mathsf{div}\,(\mathsf{A}\nabla u) + qu$ where A is determined by

$$A = \begin{bmatrix} K - K^+ & -K^+ \\ -K^+ & K - K^+ \end{bmatrix}$$

is, in fact, a 2-consistent symmetric scheme.

Proposition 4.13 *The operator L has consistent schemes of positive type iff $k_0 \geq 0$ and its matrix of leading coefficients is positive definite and strictly diagonally dominant, that is iff*

$$k_0 \geq 0 \ \text{ and } \ k_{jj} > \sum_{\substack{i=1 \\ i \neq j}}^{n} |k_{ji}|, \qquad j = 1, \ldots, n.$$

Moreover, consider $M \in O(1)$, $\phi^0, \phi^1, \Phi, h\Psi \in O(h^r)$, $r > 0$, where M, Φ are symmetric matrices and ϕ^0 is non negative when $k_0 = 0$, the function $q = k_0 - \phi^0$, the flow $\mathsf{b} = \frac{1}{2}(\hat{\mathsf{k}} - \hat{\phi}^1)$, the field of matrices A determined by

$$A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} M & M \\ M & M \end{bmatrix} - \begin{bmatrix} D & 2D \\ 0 & D \end{bmatrix} + \begin{bmatrix} \Phi + \Psi & \Psi + \Psi^t \\ 0 & \Phi + \Psi^t \end{bmatrix}$$

where D is the diagonal matrix whose non null entries are $\frac{h}{6}(h\phi_j^1 - k_j)$, $j = 1, \ldots, n$ and the difference operator $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla) + \langle \mathsf{b}, \nabla u \rangle + qu$. Then, if K is a positive definite s.d.d matrix the following properties guarantee that $\mathcal{L}_h$ is a $\min\{r, 2\}$-consistent scheme of positive type:

i) If $\lim_{h \to 0} M(h) < -K^+$ and $\lim_{h \to 0} \sum_{i=1}^{n} m_{ji}(h) > -\frac{1}{2} \sum_{i=1}^{n} k_{ji}$, $j = 1, \ldots, n$.

ii) If $M \leq -K^+$*,* $\Phi \leq 0$*,* $\Psi \leq D$ *and* $\lim_{h\to 0} \sum_{i=1}^{n} m_{ji}(h) > -\frac{1}{2}\sum_{i=1}^{n} k_{ji}$*,* $j = 1, \dots, n$*.*

Proof. As any consistent scheme of positive type is of non negative type, by using the same notations that in above proposition we get that

$$2\sum_{\substack{i=1\\i\neq j}}^{n} |k_{ij} + \Phi_{ij}| + r_j + r_{n+j} \leq 2\,(k_{jj} + \Phi_{jj}),\ \ j = 1, \dots, n.$$

Moreover, Proposition 4.1 (iii) assures that positivity implies $r_j \geq C > 0$, $j = 1, \dots, 2n$. Therefore, $\sum_{\substack{i=1\\i\neq j}}^{n} |k_{ij} + \Phi_{ij}| + C \leq k_{jj} + \Phi_{jj}$, $j = 1, \dots, n$ and the result is again obtained by taking limits when $h \to 0$.

To prove the only if condition it suffices to prove (i) or (ii). Firstly we observe that as K is s.d.d. then it is verified that $-\sum_{\substack{i=1\\i\neq j}}^{n}\Big(k_{ji} + |k_{ji}|\Big) + \sum_{i=1}^{n} k_{ji} = k_{jj} - \sum_{\substack{i=1\\i\neq j}}^{n} |k_{ji} > 0$, for all $j = 1, \dots, n$ and hence we can choose matrices $M \in O(1)$, such that $M(h) < -K^+$ and verifying the inequalities $-\sum_{i=1}^{n} k_{ji} < 2\sum_{i=1}^{n} \lim_{h\to 0} m_{ji}(h) < -\sum_{i=1}^{n}(k_{ji} + |k_{ji}|)$, $j = 1, \dots, n$.

On the other hand, under the established conditions, Proposition 4.4 (ii) assures that $\mathcal{L}_h$ is a $\min\{r, 2\}$-consistent scheme and, in addition $q \geq 0$. Therefore by Proposition 4.1 (iii), $\mathcal{L}_h$ is a positive scheme iff A is a s.d.d. M-matrix.

In both cases (i) and (ii), all off diagonal coefficients of A are non negative for h small enough and $\lim_{h\to 0} r_j(h) = \lim_{h\to 0} r_{n+j}(h) = k_{jj} + 2\lim_{h\to 0}\sum_{i=1}^{n} m_{ji}(h) > 0$, $j = 1, \dots, n$. Therefore, A is a s.d.d. M-matrix and the conclusion follows. ∎

The above proposition together with Proposition 4.8 leads to the following result.

Corollary 4.14 *Suppose that* $k_0 \geq 0$ *and* $k_{jj} > \sum_{\substack{i=1\\i\neq j}} |k_{ij}|$ *and consider* $M \in O(1)$*,* $\phi^0, \phi^1, \Phi \in O(h^r)$*,* $r > 0$*, where* M, Φ *are symmetric matrices and* ϕ^0 *is non positive when* $k_0 = 0$*, the function* $q = k_0 - \phi^0$*, the flow* $\mathsf{b} = \frac{1}{2}\,(\hat{\mathsf{k}} - \hat{\phi}^1)$*, the field of matrices* A *determined by*

$$A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} M & M \\ M & M \end{bmatrix} + \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}$$

and the difference operator $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla) + \langle \mathsf{b}, \nabla u\rangle + qu$*. Then, the following properties guarantee that* $\mathcal{L}_h$ *is a quasi-symmetric and* $\min\{r, 2\}$*-consistent scheme of positive type:*

i) If $\lim_{h\to 0} M(h) < -K^+$ *and* $\lim_{h\to 0} \sum_{i=1}^{n} m_{ji}(h) > -\frac{1}{2}\sum_{i=1}^{n} k_{ji}$, $j = 1, \dots, n$.

ii) If $M \le -K^+$, $\Phi \le 0$ *and* $\lim_{h\to 0} \sum_{i=1}^{n} m_{ji}(h) > -\frac{1}{2}\sum_{i=1}^{n} k_{ji}$, $j = 1, \dots, n$.

The above corollary together with Proposition 4.9 describe when L has isotropic and consistent schemes of positive type.

Corollary 4.15 *The differential operator* $L(u) = -a\Delta u + \langle \mathsf{k}, \nabla u\rangle + k_0 u$ *has consistent schemes of positive type iff* $a > 0$ *and* $k_0 \ge 0$*. In this case, L has isotropic consistent schemes of positive type.*

We conclude this section analyzing under what conditions $\mathcal{L}_h$ is a consistent scheme such that $-\mathsf{div}\,(\mathsf{A}\nabla u)$ is the Laplace-Beltrami operator with respect to the metric tensor A^{-1}. In this case, $\mathcal{L}_h$ is a quasi-symmetric scheme, since A is a symmetric field. Moreover Propositions 4.7 and 4.8 imply that

$$A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} M & M \\ M & M \end{bmatrix} + \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix},$$

where M and Φ are symmetric. Therefore, if $x, y \in \mathbb{R}^n$ and $z = (x,y)^t$, then

$$\langle Az, z\rangle = \langle Kx, x\rangle + \langle Ky, y\rangle + \langle Mx, x\rangle + \langle My, y\rangle + 2\,\langle Mx, y\rangle + \langle \Phi x, x\rangle + \langle \Phi y, y\rangle. \qquad (37)$$

Proposition 4.16 *The differential operator L admits consistent difference schemes $\mathcal{L}_h$ such that* A *is a metric tensor iff L is a semi-elliptic operator, that is, K is a positive semi-definite matrix.*

Proof. If L is a first order differential operator, then $K = M = 0$ and Φ is a diagonal matrix. Hence $A = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}$ and it would be enough to choose $\Phi_{jj} > 0$, so that A would be a metric tensor.

Suppose that L is a second order differential operator and consider $x \in \mathbb{R}^n$ and $z = (x, -x)^t$. Then, identity (37) implies that $\langle A(h)z, z\rangle = 2\Big(\langle Kx, x\rangle + \langle \Phi(h)x, x\rangle\Big)$ and hence $2\,\langle Kx, x\rangle = \lim_{h\to 0}\langle A(h)z, z\rangle$. So, if A is a metric tensor we conclude that $\langle Kx, x\rangle \ge 0$ and therefore K is positive semi-definite.

Conversely suppose that K is a positive semi-definite matrix and choose M symmetric and positive semidefinite and Φ symmetric and positive definite. Then, identity (37) and Cauchy-Schwarz inequality imply that

$$\langle Az, z\rangle \ge \left(\sqrt{\langle Mx, x\rangle} - \sqrt{\langle My, y\rangle}\,\right)^2 + \langle \Phi x, x\rangle + \langle \Phi y, y\rangle \ge \langle \Phi x, x\rangle + \langle \Phi y, y\rangle > 0, \quad z = (x,y)^t, \quad z \ne \mathsf{0}$$

and hence A is a metric tensor. ∎

Observe that when L is an elliptic operator we can obtain a more wide range of consistent schemes such that A is a metric tensor that those given in the above proof. Specifically, consider λ and $\hat{\lambda}(h)$ the lowest eigenvalues of K and $M(h)$ respectively. Identity (37) and Cauchy-Schwarz inequality imply again that $\langle Az, z\rangle \geq \left(\lambda + \hat{\lambda} - |\hat{\lambda}|\right)|z|^2 + \langle \Phi x, x\rangle + \langle \Phi y, y\rangle$. In conclusion, if M verifies that $\hat{\lambda} > -\dfrac{\lambda}{2}$, then A is positive definite for all h small enough.

Finally, we consider the case of isotropic schemes. In this case, by Proposition 4.9 we know that necessarily $L(u) = -a\Delta u + \langle \mathsf{k}, \nabla u\rangle + k_0 u$ with $a > 0$ and A is the field determined by

$$A = (a+\phi)\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + a_2 \begin{bmatrix} I & I \\ I & I \end{bmatrix} + a_3 \begin{bmatrix} J & J \\ J & J \end{bmatrix},$$

where $\phi \in O(h^r)$, $r > 0$ and $a_2, a_3 \in O(h^s)$, $s \in (-2, 0]$. Then A is a metric tensor iff it is true that $a_2 + n\min\{0, a_3\} > -\dfrac{a}{2}$, for h small enough. This condition had been already obtained in [2].

5 Some widely used schemes

To end the paper let us show the field of matrices and vector fields that determine the difference schemes most commonly used, see for instance [4, 12, 13]. Moreover, we will suppose from now on that the data ϕ^0, ϕ^1 and Φ are null since this hypothesis does not restrict the order of consistency of the schemes that will be considered.

Firstly we consider the first order differential operator $L(u) = \langle \mathsf{k}, \nabla u\rangle + k_0 u$, where $\mathsf{k} \neq \mathsf{0}$. By Proposition 4.3, all consistent schemes with L have the expression $\mathcal{L}_h(u) = \langle \mathsf{c}, \nabla u\rangle + qu$, where $q = k_0$, $\mathsf{c} = \mathsf{a} + \dfrac{\hat{\mathsf{k}}}{2}$ with $\mathsf{a} \in O(h^{s-1})$, $s > 0$, the homogeneous field determined by $a \in \mathbb{R}^{2n}$ where $a_j = a_{n+j}$, $j = 1, \ldots, n$. Moreover, by Proposition 4.7 $\mathcal{L}_h$ is a quasi-symmetric but not symmetric scheme which order of consistency equals $\min\{r, s, 2\}$.

In addition, L has consistent and non negative type schemes iff $k_0 \geq 0$ and applying Proposition 4.10, $\mathcal{L}_h$ is of this type iff $s \in (0, 1]$, $-\mathsf{a} \geq \frac{1}{2}|\hat{\mathsf{k}}|$. In this case the order of consistency of $\mathcal{L}_h$ is $\min\{r, s\}$ and $\mathcal{L}_h$ is not of positive type.

The natural choice, $\mathsf{a} = \mathsf{0}$, $r \geq 2$, leads to a *centered difference scheme* that is a 2-consistent scheme,

$$\mathcal{L}_h(u)(x) = \frac{1}{2h}\sum_{j=1}^{n} k_j \left(u(x_j) - u(x_{n+j})\right) + k_0\, u(x).$$

The choice, $a_j = -\frac{k_j}{2}$, $j = 1, \ldots, n$, $r \geq 1$, leads to a *backward difference scheme* that is a 1-

consistent scheme,

$$\mathcal{L}_h(u)(x) = \frac{1}{h}\sum_{j=1}^{n} k_j \left(u(x) - u(x_{n+j})\right) + k_0\, u(x).$$

The choice, $a_j = \frac{k_j}{2}$, $j = 1, \dots, n$, $r \geq 1$, leads to a *forward difference scheme* that is a 1-consistent scheme,

$$\mathcal{L}_h(u)(x) = \frac{1}{h}\sum_{j=1}^{n} k_j \left(u(x_j) - u(x)\right) + k_0\, u(x).$$

Consider now the sets $I^+ = \{j : k_j \geq 0\}$, $I^- = \{j : k_j < 0\}$ and suppose that $r \geq 1$. Then the choice $a_j = \frac{k_j}{2}$, $j \in I^+$, $a_j = -\frac{k_j}{2}$, $j \in I^-$ leads to the *upwind difference scheme* that is a 1-consistent scheme and of non negative type when $k_0 \geq 0$,

$$\mathcal{L}_h(u)(x) = \frac{1}{h}\sum_{j\in I^+} k_j \left(u(x_j) - u(x)\right) + \frac{1}{h}\sum_{j\in I^-} k_j \left(u(x) - u(x_{n+j})\right) + k_0\, u(x).$$

We consider now the difference schemes that are consistent with the second order differential operator $L(u) = -\mathsf{div}\,(K\nabla u) + \langle \mathsf{k}, \nabla u\rangle + k_0 u$, $K \neq 0$. Moreover, we will only deal with difference schemes of the greatest consistency order, that is, with order of consistency at least 2. In addition, we can look for quasi-symmetric schemes, since from Proposition 4.8 this condition does not suppose any restriction on the order of consistency, except when K is diagonal and $\mathsf{k} \neq \mathsf{0}$. On the other hand, this type of schemes are the most considered in the literature. Thus, we suppose that $\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\nabla u) + \frac{1}{2}\langle \hat{\mathsf{k}}, \nabla u\rangle + k_0 u$, where A is the field of matrices determined by

$$A = \begin{bmatrix} K+M & M \\ M & K+M \end{bmatrix}$$

with $M = (m_{ij})$ a symmetric matrix. Then, from Proposition 4.8, $\mathcal{L}_h$ is a 2-consistent quasi-symmetric scheme and when $\mathsf{k} \neq \mathsf{0}$, 2 is the greatest order of consistency for this type of schemes.

Moreover the scheme $\mathcal{L}_h$ has the following expression that corresponds to replace all derivatives by centered differences,

$$\begin{aligned}\mathcal{L}_h(u)(x) = {} & \frac{1}{h^2}\sum_{j=1}^{n}\Big(\sum_{i=1}^{n}(k_{ij} + 2m_{ij})\Big)\Big(2u(x) - u(x_j) - u(x_{n+j})\Big) \\ & - \frac{1}{h^2}\sum_{1\leq i<j\leq n}^{n} m_{ij}\Big(2u(x) - u(x_{ij}) - u(x_{n+in+j})\Big) - \frac{1}{h^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(k_{ij} + m_{ij})\Big(u(x) - u(x_{jn+i})\Big) \\ & - \frac{1}{2h^2}\sum_{j=1}^{n} m_{jj}\Big(2u(x) - u(x_{jj}) - u(x_{n+jn+j})\Big) + \frac{1}{2h}\sum_{j=1}^{n} k_j\Big(u(x_j) - u(x_{n+j})\Big) + k_0 u(x).\end{aligned}$$

In particular, the choice $M=0$ leads to a generalization of the standard difference scheme, see for instance [13, pp. 286-288],

$$\begin{aligned}\mathcal{L}_h(u)(x) &= \frac{1}{h^2}\sum_{j=1}^{n}\Big(\sum_{i=1}^{n}k_{ij}\Big)\Big(2u(x)-u(x_j)-u(x_{n+j})\Big)\\ &-\frac{1}{h^2}\sum_{1\le i<j\le n}^{n}k_{ij}\Big(2u(x)-u(x_{jn+i})-u(x_{in+j})\Big)\\ &+\frac{1}{2h}\sum_{j=1}^{n}k_j\Big(u(x_j)-u(x_{n+j})\Big)+k_0u(x),\end{aligned}\tag{38}$$

that is of positive type iff $k_0\ge 0$ and K is a symmetric and s.d.d. M-matrix.

Other difference schemes that one can find in the literature correspond to the choice $m_{jj}=0$ and $m_{ij}=-k_{ij}$, $i,j=1,\dots,n$, $i\ne j$,

$$\begin{aligned}\mathcal{L}_h(u)(x) &= \frac{1}{h^2}\sum_{j=1}^{n}\Big(k_{jj}-\sum_{\substack{i=1\\ i\ne j}}^{n}k_{ij}\Big)\Big(2u(x)-u(x_j)-u(x_{n+j})\Big)\\ &+\frac{1}{h^2}\sum_{1\le i<j\le n}^{n}k_{ij}\Big(2u(x)-u(x_{ij})-u(x_{n+in+j})\Big)\\ &+\frac{1}{2h}\sum_{j=1}^{n}k_j\Big(u(x_j)-u(x_{n+j})\Big)+k_0u(x),\end{aligned}\tag{39}$$

and to the choice $m_{jj}=0$, $m_{ij}=-\dfrac{1}{2}\Big(\dfrac{k_{jj}}{n-1}+k_{ij}\Big)$, $i,j=1,\dots,n$, $i\ne j$,

$$\begin{aligned}\mathcal{L}_h(u)(x) &= \frac{1}{2\,h^2}\sum_{1\le i<j\le n}^{n}\Big(\frac{k_{jj}}{n-1}-k_{ij}\Big)\Big(4u(x)-u(x_{ij})-u(x_{n+in+j})-u(x_{jn+i})-u(x_{in+j})\Big)\\ &+\frac{1}{2h}\sum_{j=1}^{n}k_j\Big(u(x_j)-u(x_{n+j})\Big)+k_0u(x).\end{aligned}\tag{40}$$

Moreover, the scheme (39) is of positive type iff $k_0\ge 0$ and K is a non negative and s.d.d. matrix, whereas the scheme (40) is of positive type iff $k_0\ge 0$ and $k_{jj}\ge (n-1)\,|k_{ij}|$, $1\le i<j\le n$, which in particular implies that K is a s.d.d. matrix.

To finish we consider the case in which K is a diagonal matrix. Then, the schemes (38) and (39) coincide and determine the well-known 2-consistent scheme which is the unique first order difference operator of this type and the scheme (40) is the *cross scheme*.

If we consider $m_{jj}=0$, $m_{ij}=-\dfrac{k_{ii}+k_{jj}}{12(n-1)}$, $i,j=1,\dots,n$, $i\neq j$, we get the scheme

$$\begin{aligned}\mathcal{L}_h(u)(x) &= \frac{1}{6\,h^2}\sum_{j=1}^{n}\Big(5\,k_{jj}-\frac{1}{n-1}\sum_{\substack{i=1\\ i\neq j}}^{n}k_{ii}\Big)\Big(2u(x)-u(x_j)-u(x_{n+j})\Big)\\ &+\frac{1}{12\,h^2}\sum_{1\le i<j\le n}^{n}\Big(\frac{k_{ii}+k_{jj}}{n-1}\Big)\Big(4u(x)-u(x_{ij})-u(x_{n+in+j})-u(x_{jn+i})-u(x_{in+j})\Big)\\ &+\frac{1}{2h}\sum_{j=1}^{n}k_j\Big(u(x_j)-u(x_{n+j})\Big)+k_0u(x),\end{aligned}$$

that is a generalization of the *nine-point scheme* for the bidimensional Laplace operator, see [4]. If s is such that $k_{ss}=\min\limits_{j=1,\dots,n}\{k_{jj}\}$, then the above scheme is of positive type iff $k_0\ge 0$, $k_{jj}\ge 0$, $j=1,\dots,n$, and $5k_{ss}>\dfrac{1}{n-1}\sum\limits_{\substack{i=1\\ i\neq s}}^{n}k_{ii}$.

Finally, when $\mathsf{k}=0$, the choice $M=\frac{1}{6}\,K$ leads to the 4-consistent scheme, see newly [4],

$$\begin{aligned}\mathcal{L}_h(u)(x) &= \frac{4}{3h^2}\sum_{j=1}^{n}k_{jj}\Big(2u(x)-u(x_j)-u(x_{n+j})\Big)\\ &-\frac{1}{12h^2}\sum_{j=1}^{n}k_{jj}\Big(2u(x)-u(x_{jj})-u(x_{n+jn+j})\Big)-k_0u(x)\end{aligned}$$

that is not of non negative type.

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